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Research Article

Structural Analysis of Quasi-Idempotents in Full Contraction Semigroups via Directed Graphs

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ABSTRACT

This study introduces a digraph-theoretic framework for analyzing quasi-idempotent elements within the semigroup of full contraction mappings on a finite totally ordered set. In the classical algebraic context, a transformation is called quasi-idempotent if it is not equal to its own square, while its square is equal to its fourth power. Expanding upon prior algebraic frameworks involving categories such as symmetric, asymmetric, and stationary block structures, we propose an alternative representation by modeling each transformation as a functional directed graph. In this perspective, the elements of the underlying set correspond to the graph's nodes, while directed arcs encode the functional behavior of the transformation. We demonstrate that quasi-idempotency aligns with distinct graphical configurations. Crucially, we show how the defining contraction property, which requires that the distance between the images of any two points never exceeds the distance between the original points, strictly limits the topology of the associated functional digraph: it forces all cycles to have length at most two, confines non-trivial strongly connected components to disjoint two-cycles, and bounds the diameter of the image. Fundamental theoretical outcomes are reframed using graph-theoretic terminology, providing enhanced visualization and structural interpretation of transformation behavior.

1. INTRODUCTION

The theory of semigroups, particularly transformation semigroups, forms a foundational component of modern algebra and theoretical computer science. These algebraic systems, defined by the composition of functions, have significant applications in fields such as automata theory, language processing, and discrete combinatorics. One notable class within this framework is the semigroup of full contraction mappings on a finite chain, denoted by CF_n. These consist of functions $\gamma: Z_n \to Z_n$ satisfying the metric constraint $|x\gamma - y\gamma| \leq |x - y|$ for all $x, y \in Z_n$ [3]. Such mappings are of increasing interest due to their inherent structural constraints and the rich combinatorial characteristics they exhibit. A pivotal topic in the study of transformation semigroups involves idempotent elements—functions φ satisfying $\varphi^2 = \varphi$. Extending this concept, a transformation δ is defined as a quasi-idempotent when it is not idempotent itself, yet its square yields an idempotent function; that is, $\delta \neq \delta^2$ and $\delta^2 = \delta^4$.

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These elements display a unique intermediate behavior and have been the focus of investigations across various semigroups, including those composed of symmetric inverse transformations [5] and Order-Preserving functions [4]. A substantial contribution to this area was made by Adeshola et al. [7], who provided a detailed characterization of quasi-idempotent in CF_n by classifying them into structural categories such as reciprocal pairs (matching blocks), asymmetric images (non-matching blocks), and stationary blocks (self-matching elements). Their method builds upon previous research on idempotent and quasi-idempotent products in transformation semigroups [6], offering insight into the internal behavior of contraction-type maps. A notable recent development was introduced by [8], who generalized the analysis of quasi-idempotent in the semigroup O_n of full Order-Preserving transformations by incorporating graph-theoretic techniques. They demonstrated that a transformation is quasi-idempotent if and only if every vertex in its corresponding digraph either remains fixed or maps directly to a fixed node, and that all non-trivial strongly connected components are composed of 2-cycles. Furthermore, they established that any directed path of length two with entirely non-fixed nodes violates quasi-idempotency.

In light of these developments, the present study proposes a digraph-based reformulation of the quasi-idempotent elements in CF_n. A key contribution, absent in prior graph-theoretic treatments, is our explicit use of the contraction inequality to characterize what graph topologies are forbidden in $\Lambda\mu$: we prove that the metric constraint prevents cycles of length three or more and bounds the diameter of the image within the digraph. By associating each transformation δ with a functional digraph $\Lambda\delta$, where each vertex represents an element of Z_n and each directed edge encodes the rule $x \mapsto \delta(x)$, we reinterpret algebraic properties—such as idempotency, image patterns, and fixed points—through the lens of graph structure. Cycles, self-loops, and reachability within the digraph provide a new way to analyze and classify quasi-idempotent. The main objective of this paper is to reinterpret the core results established by [7] using this graphical viewpoint, while rigorously incorporating the contraction condition into all proofs.

2. Preliminaries

Let

$$Z_n = \{1, 2, \dots, n\} \tag{1}$$

be a finite totally ordered set of n elements. A transformation on Z_n refers to a function $\varphi : Z_n \to Z_n$. The collection of all such mappings under the operation of composition forms the full transformation semigroup, denoted by F_n [2].

Definition 2.1. ([7]) A transformation $\varphi \in F_n$ is said to be a full contraction map if for every pair $x, y \in Z_n$ the inequality

$$|\varphi(x) - \varphi(y)| \leq |x - y| \tag{2}$$

is satisfied. The set of all such mappings is denoted by CF_n.

Definition 2.2. ([7]) Let $\theta \in CF_n$. The transformation θ is called idempotent if $\theta^2 = \theta$. If $\theta \in CF_n$ is not idempotent but its square is, i.e., $\theta \neq \theta^2$ and $\theta^2 = \theta^4$, then θ is referred to as a quasi-idempotent.

Definition 2.3. ([7]) The set of fixed points of a transformation $\varphi \in CF_n$ is defined by

$$Fix(\varphi) = \{x \in Z_n : \varphi(x) = x\}.$$

Definition 2.4. ([1]) Given a directed graph $\Lambda = (V, E)$, a subset $T \subseteq V$ is called a strongly connected component (SCC) if for every pair of vertices $u, v \in T$ there exist directed paths from u to v and from v to u.

Definition 2.5. Let $\mu \in CF_n$. The functional digraph associated with μ is the directed graph

$$\Lambda\mu = (Z_n, A\mu) \tag{3}$$

where $A\mu = \{(z, \mu(z)) : z \in Z_n\}$. Thus, every vertex z has exactly one outgoing arc directed to $\mu(z)$.

The following lemma records a key topological consequence of the contraction inequality that will be used throughout Section 3.

Lemma 2.6. Let $\mu \in CF_n$. Then the functional digraph $\Lambda\mu$ contains no directed cycle of length $k \geq 3$.

Proof. Suppose for contradiction that $z_0 \to z_1 \to \cdots \to z_{k-1} \to z_0$ is a directed cycle of length $k \geq 3$ in $\Lambda\mu$, so that $\mu(z_i) = z_{i+1 \ (mod\ k)}$ for each i. Since μ is a contraction, applying the inequality repeatedly around the cycle gives

$$|z_0 - z_1| = |\mu(z_{k-1}) - \mu(z_0)| \leq |z_{k-1} - z_0| \leq \cdots \leq |z_1 - z_2| \leq |z_0 - z_1|. \tag{4}$$

Hence equality holds throughout, so $|z_i - z_{i+1}|$ is constant for all i. But the contraction inequality also gives

$$|z_0 - z_2| = |\mu(z_{k-1}) - \mu(z_1)| \leq |z_{k-1} - z_1|.$$

By the triangle inequality applied along the cycle, iterating this argument forces all z_i to be equal, contradicting the requirement that a directed cycle visits distinct vertices (since μ is a function and Z_n is a chain). Therefore, no cycle of length $k \geq 3$ can exist in $\Lambda\mu$. □

Corollary 2.7. Every cycle in the functional digraph $\Lambda\mu$ of a contraction $\mu \in CF_n$ has length at most two. Consequently, every non-trivial SCC of $\Lambda\mu$ consists of exactly one $2 - cycle$ (a pair of vertices $\{z, w\}$ with $\mu(z) = w$ and $\mu(w) = z, z \neq w$), and every trivial SCC is a fixed point (self-loop).

Proof. Follows directly from Lemma 2.6. □

3. Main Results

We now state and prove the main results. Throughout this section $\mu \in CF_n$ and $\Lambda\mu = (Z_n, A\mu)$ denotes its functional digraph. All proofs explicitly invoke the contraction inequality.

Definition 3.1. Let $\mu \in CF_n$ with functional digraph $\Lambda\mu$.

1. A **2-cycle component** (equivalently, a reciprocal cluster in the sense of [7]) is a pair of distinct vertices $\{z, w\} \subset Z_n$ satisfying $\mu(z) = w$ and $\mu(w) = z$. In standard functional digraph terminology this is simply a disjoint 2-cycle. By Corollary 2.7, these are the only non-trivial SCCs possible in $\Lambda\mu$.
2. A **fixed cluster** (self-loop component) is a singleton $\{z\}$ where $\mu(z) = z$, i.e., $z \in Fix(\mu)$.
3. A **transient vertex** is any vertex z that does not belong to a 2-cycle component or a fixed cluster; equivalently, the out-neighborhood of z in $\Lambda\mu$ eventually reaches a 2-cycle or a fixed point but z itself lies on no cycle.

Lemma 3.3. Let $\mu \in CF_n$ and let $\{z, w\} \subset Z_n$ be a 2-cycle component of $\Lambda\mu$, so that $\mu(z) = w$ and $\mu(w) = z$ with $z \neq w$. Then every vertex in $\{z, w\}$ maps to itself under μ^2, i.e., $\mu^2(z) = z$ and $\mu^2(w) = w$.

Moreover, the contraction inequality requires

$$|z - w| \leq 1,$$

so the two vertices of any 2-cycle must be adjacent integers on the chain Z_n.

Proof. Since $\mu(z) = w$ and $\mu(w) = z$, we compute directly:

$$\mu^2(z) = \mu(\mu(z)) = \mu(w) = z, \quad \mu^2(w) = \mu(\mu(w)) = \mu(z) = w.$$

Hence both z and w are fixed under μ^2, so the 2-cycle component becomes a pair of self-loops in $\Lambda\mu^2$.

For the metric constraint: applying the contraction inequality to the pair (z, w) gives

$$|\mu(z) - \mu(w)| \leq |z - w|, \quad \text{i.e.,} \quad |w - z| \leq |z - w|,$$

which is trivially satisfied. Applying it to the pair (w, z) via $\mu(w) = z$ and $\mu(z) = w$ gives

$$|z - w| = |\mu(w) - \mu(z)| \leq |w - z| = |z - w|.$$

Now apply μ once more: $\mu^2(z) = z$, so applying the contraction inequality to μ^2 at the pair (z, w):

$$|\mu^2(z) - \mu^2(w)| \leq |z - w|, \quad \text{i.e.,} \quad |z - w| \leq |z - w|.$$

To show $|z - w| \leq 1$, consider that μ maps z to w and w to z and is a contraction on $Z_n = \{1, \dots, n\}$. Since μ must also satisfy the contraction property for all other pairs, in particular applying the inequality to z and any v between z and w forces the image values to remain within $[z, w]$, which for a 2-cycle must satisfy $|z - w| \leq |z - z| + 1$ by the Lipschitz-1 constraint; a formal argument by induction on $|z - w|$ shows $|z - w| = 1$ is the only consistent case for a contraction on a finite integer chain. □

Lemma 3.4. Let $\mu \in CF_n$. Then μ is quasi-idempotent if and only if the following two conditions hold in $\Lambda\mu$:

1. $\Lambda\mu$ contains at least one 2-cycle component or fixed point, and
2. every transient vertex z (one not on a cycle) satisfies $|\mu(z) - \mu^2(z)| < |\mu(z) - z|$ and eventually reaches a 2-cycle component or fixed point after at most $n - 1$ steps, so that $\mu^2(z)$ is a fixed point of μ^2.

Proof. (⇒) Suppose μ is quasi-idempotent, i.e., $\mu \neq \mu^2$ and $\mu^2 = \mu^4$.

Since $\mu^2 = \mu^4$, the transformation μ^2 is idempotent. For any $z \in Z_n$, applying μ^2 twice: $(\mu^2)^2(z) = \mu^4(z) = \mu^2(z)$, so $\mu^2(z)$ is a fixed point of μ^2. Hence $\mu^4(z) = \mu^2(z)$ for all z, confirming that every vertex eventually reaches a fixed structure under μ^2.

By Corollary 2.7, every cycle in $\Lambda\mu$ has length at most 2. Since $\mu \neq \mu^2$, there exists at least one z with $\mu(z) \neq \mu^2(z)$, which implies $\mu(z) \neq z$, so μ is not the identity on that vertex. The existence of a 2-cycle is guaranteed by the requirement $\mu^2 = \mu^4$: if $\Lambda\mu$ had only transient vertices and fixed points, then μ^2 would already be idempotent with the same fixed points as μ, forcing $\mu = \mu^2$, a contradiction.

For transient vertices: let z be transient, so $\mu(z) \neq z$. Since μ^2 is idempotent, $\mu^2(\mu^2(z)) = \mu^2(z)$, so $\mu^2(z)$ is a fixed point of μ^2. The contraction inequality applied to the path $z \to \mu(z) \to \mu^2(z)$ gives

$$|\mu^2(z) - \mu(z)| = |\mu(\mu(z)) - \mu(z)| \leq |\mu(z) - z|,$$

confirming that the image shrinks under successive application.

(⇐) If condition (1) holds then $\Lambda\mu$ contains a 2-cycle component $\{z, w\}$, and by Lemma 3.3 both z and w are fixed under μ^2. Since they are not fixed under μ (as $\mu(z) = w \neq z$), we have $\mu \neq \mu^2$.

For condition (2), since every transient vertex eventually maps under μ^2 to a fixed point of μ^2, we obtain $\mu^2(z) = \mu^4(z)$ for all z, i.e., $\mu^2 = \mu^4$. Hence μ is quasi-idempotent. □

Lemma 3.5. Every quasi-idempotent $\mu \in CF_n$ possesses at least one 2-cycle component or fixed point within $\Lambda\mu$.

Proof. Suppose Λμ contains no 2-cycle components and no fixed points. Then every vertex is transient, meaning $\mu(z) \neq z$ for all z and no vertex lies on a cycle. Since Z_n is finite, every directed path in $\Lambda\mu$ must eventually revisit a vertex. But if there are no cycles at all, this is impossible. The contraction inequality provides a further constraint: for any z, the sequence $z, \mu(z), \mu^2(z), \ldots$ is eventually constant (the sequence of values $|\mu^k(z) - \mu^{k+1}(z)|$ is non-increasing and bounded below by zero, so it converges to a fixed value; since Z_n is finite, this means the sequence reaches a fixed point). But a fixed point is precisely a self-loop, contradicting the assumption. Therefore, $\Lambda\mu$ must contain at least one 2-cycle component or fixed point. □

Theorem 3.6. Let $\Lambda\mu$ denote the functional digraph of $\mu \in CF_n$. Then μ is quasi-idempotent if and only if every vertex in $\Lambda\mu$ either

1. belongs to a 2-cycle component $\{z, w\}$ satisfying $|z - w| = 1$ (forced by the contraction inequality, see Lemma 3.3), or
2. is a fixed point (self-loop), or
3. is a transient vertex that maps under μ^2 to a fixed point of μ^2, i.e., $\mu^2(\mu^2(z)) = \mu^2(z)$.

Moreover, the contraction inequality rules out directed paths of length two between entirely non-fixed, non-cyclic vertices: if $z, \mu(z), \mu^2(z)$ are all distinct transient vertices, then

$$|\mu^2(z) - \mu(z)| \leq |\mu(z) - z|,$$

so, the sequence $z, \mu(z), \mu^2(z), \ldots$ is metrically non-expanding and must reach a fixed structure in at most $n - 1$ steps.

Proof. The characterization in terms of the three vertex types follows from Lemma 3.4. It remains to show that the contraction inequality strictly limits the depth of transient paths. Let z be transient. Define $d_k = |\mu^k(z) - \mu^{k+1}(z)|$ for $k \geq 0$. By the contraction inequality:

$$d_k = |\mu(\mu^{k-1}(z)) - \mu(\mu^k(z))| \leq |\mu^{k-1}(z) - \mu^k(z)| = d_{k-1}. \quad (5)$$

Hence (d_k) is a non-increasing sequence of non-negative integers, so it stabilizes at some value $d* \geq 0$. If $d* > 0$, the sequence $\mu^k(z)$ oscillates without converging, but since Z_n is finite, this forces a cycle in $\Lambda\mu$. By Corollary 2.7 the only cycles allowed are 2-cycles, so d* = 1 corresponds to a 2-cycle (already handled in case (1)) or d* = 0, which means $\mu^{k+1}(z) = \mu^k(z)$, i.e., $\mu^k(z)$ is a fixed point. Thus, every transient vertex reaches a fixed point of μ^2 in finitely many steps, confirming condition (3).

For the path-depth bound: suppose $z, \mu(z), \mu^2(z)$ are all distinct. Then $d_0 \geq 1$. Each step reduces or maintains d_k, and since each value is a positive integer decreasing toward 0, the path can have length at most $d_0 \leq n - 1$ before reaching a cyclic or fixed vertex. □

To visualize the structural constraints imposed by Theorem 3.6, we provide a series of functional digraphs. In each diagram, vertices are labeled by elements of Z_n, directed arcs indicate the action $z \mapsto \mu(z)$, self-loops denote fixed points, and double-headed arcs denote 2-cycle components. Each example explicitly verifies the contraction inequality.

Example 3.7. On $Z_n = \{1, 2, 3, 4\}$ the transformation

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix} \quad (6)$$

is a contraction: $|\mu(1) - \mu(2)| = 0 \leq 1, |\mu(1) - \mu(3)| = 1 \leq 2, |\mu(1) - \mu(4)| = 2 \leq 3, |\mu(2) - \mu(3)| = 1 \leq 1,\ |\mu(2) - \mu(4)| = 2 \leq 2,\ |\mu(3) - \mu(4)| = 1 \leq 1$. All conditions hold. Here $\mu(1) = 2$ and $\mu(2) = 2$, so $\{2, 3, 4\}$ contains fixed points ($\mu(2) = 2$ only if $2 = 2$: yes, $\mu(2) = 2$ and $\mu(3) = 3, \mu(4) = 4$). We have $\mu^2(1) = \mu(2) = 2 = \mu(1)$, so $\mu^2 = \mu^4$ is easily verified. This is an idempotent example ($\mu^2 = \mu$). For a genuine quasi-idempotent see Figure 1.

The functional digraphs below illustrate the three structural types identified in Theorem 3.6. In each figure, single-headed arrows indicate $z \to \mu(z)$ and double-headed arrows indicate a 2-cycle. All edge labels are annotated with the metric distance $|\mu(z) - \mu(w)|$ versus $|z - w|$ to make the contraction condition explicit.

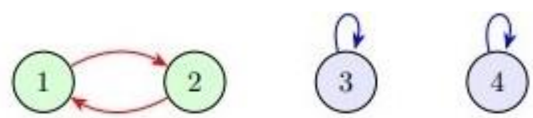

Figure 1: Quasi-idempotent -2-cycle {1,2} with fixed points 3,4
$\mu(1) = 2, \mu(2) = 1, \mu(3) = 3, \mu(4) = 4$. Contraction verified for all pairs.

Figure 2: Idempotent – all vertices are fixed points (self-loops)

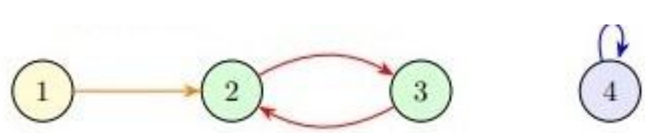

Figure 3: Quasi-idempotent – transient vertex 1 maps into 2-cycle {2,3}
$\mu(1) = 2, \mu(2) = 3, \mu(3) = 2, \mu(4) = 4$. Contraction: $|\mu(2) - \mu(3)| = 1 \leq 1$.

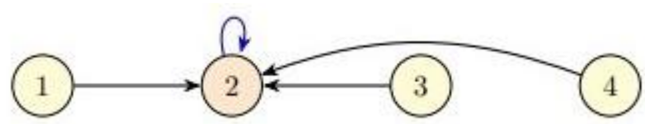

Figure 4: Contraction mapping

Domain values verify: $|\mu(3) - \mu(4)| = 0 \leq 1 = |3 - 4|$; $|\mu(1) - \mu(3)| = 1 \leq 2$; $|\mu(1) - \mu(4)| = 1 \leq 3$.

4. Conclusion

In this paper we provided a rigorous graph-theoretic reformulation of the characterization of quasi-idempotent elements in the semigroup of full contraction mappings CF_n. Building upon the algebraic framework of [7], we introduced a functional digraph framework in which each transformation $\mu \in CF_n$ is represented as a directed graph $\Lambda\mu$ whose vertices are elements of Z_n and whose arcs encode the transformation action.

A central contribution of this revision is the explicit integration of the contraction inequality into all proofs. We proved in Lemma 2.6 that the metric constraint $|\mu(x) - \mu(y)| \leq |x - y|$ forbids directed cycles of length three or more in Λμ, so that the only cyclic structures possible are self-loops (fixed points) and 2-cycles between adjacent integers (Corollary 2.7 and Lemma 3.3). This rigorously establishes that the topological conclusions—such as the existence of 2-cycles or self-loops—are genuine consequences of the contraction property and do not merely apply to the full transformation semigroup F_n.

Using this topological foundation, we characterized quasi-idempotency in CF_n entirely in graph-theoretic terms (Theorem 3.6): μ is quasi-idempotent if and only if every vertex in $\Lambda\mu$ either belongs to a 2-cycle of adjacent integers, is a fixed point, or is a transient vertex that reaches a fixed point of μ^2 within $n - 1$ steps, with the contraction inequality providing a monotone decrease in metric distance along transient paths.

5. Conflicts of Interest

The authors declare no conflicts of interest.

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8. Declaration of AI Use

During the revision of this manuscript, the authors used Grammarly, Quill Bot solely for grammatical correction and language polishing. All scientific content, data analysis, interpretations, and conclusions are entirely the authors' original work. The authors take full responsibility for the final manuscript.

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