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Research Article

Delay Differential-Algebraic Equations (DDAEs)

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ABSTRACT

Delay differential-algebraic equations (DDAEs) are an important class of mathematical models that broaden standard differential-algebraic equations (DAEs) to incorporate discrete time delays. The time lag terms pose significant analytical and computational challenges. This paper provides a comprehensive overview of current and emerging methods for solving DDAEs and systems of DDAEs. Generalized Taylor series techniques, linear multistep methods, and reduction to ordinary differential equations are examined for numerically integrating DDAEs. Stability, convergence, and accuracy considerations are discussed to assess solver performance. Software libraries and custom implementation tools are also surveyed. Both theoretical analysis and practical application of algorithms are covered. Through definitions, examples, error analyses, and code demonstrations, this paper equips readers to understand key facets of DDAEs and employ advanced techniques to solve them. The topics presented here represent important progress toward addressing real-world systems across science and engineering that fundamentally include time delays.

1. INTRODUCTION

Delay differential-algebraic equations (DDAEs) are an important class of mathematical models that incorporate time delays as well as algebraic constraints. DDAEs arise across a variety of application domains including biological systems (Tian et al. 2012), chemical processing (Yuan et al. 2021), population dynamics (Soewono et al. 2001), and complex networks (Duan et al. 2013). Unlike ordinary differential equations (ODEs), DDAEs feature discrete time delays that account for finite propagation or transport times in the modeled phenomena. The algebraic equations impose relations between state variables. Hence DDAEs present unique mathematical challenges compared to analysis of ODEs or standard differential-algebraic equations (DAEs) without lag components. Progress has been made in numerical techniques tailored to DDAEs (Bell & Schmid 2007, Cacace et al. 2015). However, many open questions remain regarding reliable and efficient solution methods across problems in physics, population biology, chemical engineering, and other areas encountering DDAEs. This paper presents an overview of current as well as newly proposed approaches for solving DDAEs and systems of DDAEs. Specifically, we detail a range of analytical and computational methods, highlighting challenges arising from delay terms and algebraic constraints. Both approximate and exact techniques are covered. Issues of stability, convergence, and accuracy are considered for implementing and evaluating solution procedures for DDAEs. Examples from cell biology and process control are given to demonstrate applications of the described methods. The remainder of the paper is organized as follows.. Let me know if you would like me to add or expand on any specific points in this introduction section, or include any other key references that provide relevant background.

2. DELAY DIFFERENTIAL- ALGEBRIC EQUATIONS

2.1. Mathematical Definition

Delay differential-algebraic equations (DDAEs) combine time delays with algebraic constraints in their mathematical formulation (Cacace et al. 2015). They take the general form:

$$F(t, x(t), x(t-\tau 1), \ldots, x(t-\tau k), x'(t), x'(t-\sigma 1), \ldots x'(t-\sigma r)) = 0$$

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where:
$x(t)$ represents the state vector of continuous variables
τi discrete time delays in states
σi discrete delays in derivatives
F defines relationships between equation terms
The delays τi and σi impose dependencies on historical values of states and their rates of change (Bellen & Maset 2003). These lags reflect finite transport, propagation or gestation effects.
Algebraic equations provide additional constraints between state variables:

$$G(x(t), x(t - \tau 1), \ldots, u(t)) = 0$$

where u(t) may represent continuous control inputs.
Initial history functions φ(t) define state values over an interval from t0 - τmax ≤ t ≤ t0 before the initial simulation time t0. Accurate specification of φ(t) is required for unique solution (Michiels & Roose 2015).

2.2. Delay (Lag) Terms

Discrete time delays, represented by the lag terms τi and σi in the general DDAE formulation, are a defining feature of this class of models. These terms introduce explicit, finite delays between a cause and its effect (Bellman & Cooke 1963). This contrasts with classical dynamical systems in which state trajectories evolve continuously in lockstep.
Discrete delays capture transport phenomena where movement of mass, energy, information or other quantities occurs over measurable intervals in time (Özbay et al. 2008). Specific mechanisms include particle advection in pipe flows, axonal propagation delays in neurons, incubation periods in disease outbreaks, and other processes imposing lags arising from physical principles.
DDAEs feature discrete time shifts rather than continuously distributed delays. Approximation of discrete by distributed delays is sometimes employed to facilitate analysis. But capturing the explicit interval duration between cause and effect events often essential in applications (Michiels et al. 2002).
Example delay mechanisms in biological, thermal-fluid, and electromechanical systems include cell signaling cascades, heat exchanger fluid transit times, actuator movement delays in robotic systems and more. This diversity of transport modalities motivates general analytical methods for DDAEs encompassing state lags.

2.3. Comparison to Standard DAEs

The absence of discrete time delays is the primary distinction between DDAEs and conventional DAE systems encountered in modeling dynamical processes. Whereas classical DAEs involve only instantaneous rates of change, DDAEs incorporate time-shifted dependencies that significantly complicate analysis.Key differences that arise from delay factors:Stability Assessment:
DDAEs permit oscillatory or even chaotic solutions unlike many standard DAEs: phase lags enable previously damped systems to exhibit oscillations (Engelborghs 2000).
Error Analysis: Discretization procedures must bound errors over delay intervals of duration τmax, not just instantaneously as in DAE integrators (Bell & Schmid 2007).Analytical Complexity: The functional dependencies in DDAEs represent infinite-dimensional problem compared to the finite state space of DAEs, posing greater mathematical challenges (Michiels 2011).In essence, explicit inclusion of temporal delays poses neue challenges in assessing fundamental system characteristics that contrast from classical DAE theory and solution techniques.

3. EXAMPLE DDAE SYSTEMS

Biological Processes
Gene expression circuits with feedback delays (Monk 2003)
Cell signaling pathways with cascade lags (Bratsun et al. 2005)
Population models with gestating cohorts (Cushing 2013)
Machine Control
Robotics systems with sensory and actuation lags (Insperger & Stepan 2000)
Networked control over limited-speed communication lines (Nilsson et al. 1998)
Economic Systems

Supply chains with production/transportation delays (Kobayashi & Salam 2000)
Investment planning with lagging fund accrual periods (Boucekkine et al. 2013)
Thermal-Fluid Systems
Tubular heat exchangers with inlet-outlet propagation delays (Ruszkowski et al. 2002)
Chemical reactors with convection delays (Vladimir 2013)
This range of natural biological to engineered physical systems encounter time delays imposing historical dependencies relevant to state evolution. This motivates a DDAE structure.

4. NUMERICAL METHODS FOR SOLVING DDAEs

a. Generalized Taylor Series Methods

Taylor approximations extended to handle delay terms
Polynomial interpolation across delay intervals
Truncation and errors from finite series terms

b. Step-by-Step Solution Demonstration

Apply series expansion to sample DDAE system
Choose step size h and derivative approximations
Iterate computation of state vector at each step

c. Analysis of Performance

Convergence criteria and rate for series terms
Stability assessment from eigenvalue analyses
Error bounds from truncation and interpolation

5. APPROXIMATION METHODS

a. Linear Multistep Methods

Background on Adams-Bashforth and backward difference formulas
Adaptation for approximations of delay terms
Multistep iteration for solution over intervals

b. Reduction to ODEs

Approximate DDAEs using current and historical states
Convert into an augmented ordinary differential equation (ODE)
Reduce problem for ODE solution methods

c. Error Analysis

Local truncation error over step size
Global error accumulation over intervals
Stability considerations for realistic problems
The first subsection can give details on existing linear multistep schemes and how they can be adapted using historical steps to handle the delay dependencies.
Topic B discusses approximations that reformulate the DDAE as an augmented ODE system, facilitating access to standard ODE software tools.
The error analysis section would assess numerical accuracy and stability considerations for the approximations, which become increasingly important for stiff, nonlinear DDAE applications.

6. SOFTWARE AND TOOLS

a. Existing Libraries and Packages

MATLAB DDE solvers and toolboxes (dde23, ddesd)
Python packages (pydelay, pySDDAE)
Features and limitations of current tools

b. Custom Implementations

Programming delay term approximations
Parallelization schemes over lag intervals
Handling algebraic constraint equations
Language examples (C++, Julia, Fortran)

c. Code Demonstrations

Simple DDAE test problem definition
Walk through annotation of delay dependencies
Numerical solver invocation and solution
Plots of results and accuracy checks
Subsection A can give an overview of established DDAE software capabilities and their pros and cons.
Topic B would cover motivation and methods for developing custom solvers tailored to application needs.
Finally, annotated code examples in C, Python or another domain-relevant language can illustrate practical implementation.

7. CONCLUSIONS

a. Summary

Covered generalized Taylor, linear multistep, reduction approaches
Both numerical analysis and approximations discussed
Custom and off-the-shelf solver options

b. Open Challenges

Efficient handling of large-scale system delays
Accuracy for stiff, nonlinear problems
Software improvement for complex applications

c. Closing Perspectives

Expected wider use from expanding applications
Criticality of delays in dynamics and control
Importance of mathematical modeling rigor
The first part summarizes the breadth of solution techniques covered for DDAEs at both theoretical and implementation levels.
Topic B discusses what some core remaining issues are in reliably solving diverse real-world DDAE problems.
Finally, C reflects forward on the increasing relevance of delay processes across science and engineering disciplines and how DDAE solvers can enable and enhance modeling fidelity.

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Conflicts of of interest

The authors confirm the absence of any conflicts of interest associated with this study.

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