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Research Article

Variational Iteration Approach for Solving Fractional Integro-Differential Equations with Conformable Differintegrals

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ABSTRACT

Fractional integro-differential equations involving conformable differintegrals have promising applications but lack solution methods. This work develops a variational iteration approach for such equations. Conformable fractional derivatives and integrals that generalize integer-order equivalents are defined. Existence and uniqueness results are established for a class of nonlinear fractional integro-differential equations. These exist under less restrictive smoothness assumptions than integer-order cases. The variational iteration method (VIM) is then applied, providing an analytical approximate technique for integro-differential problems of fractional order. Convergence analysis demonstrates the efficiency and accuracy of the VIM solutions. Several test cases validate the VIM, matching analytical solutions available for simpler fractional differential sub-cases. The proposed technique advances available methods for this emerging class of fractional integro-differential equations. Significantly, it enables application of such models by allowing accurate solution of associated mathematical system representations. Extensions to include more singular equation classes, comparisons with other methods, and real-world applications are suggested as future work.

The abstract summarizes the key points: significance of problem, fractional models used, existence/uniqueness proofs, VIM approach and analysis, test case validations, implications for applications

1. INTRODUCTION

Fractional calculus has become a growing field of mathematical analysis, with differential and integral operators generalized to non-integer orders. This allows modeling of complex dynamics and heterogeneity in ways not captured by classical integer-order models [1]. Applications are numerous, including viscoelasticity [2], electrochemistry [3], and nonlinear oscillations [4]. Fractional derivatives enable memory and hereditary properties of various processes to be mathematically described. Recent years have seen fractional modeling expanded to include differential equations with integral terms as well. These fractional integro-differential equations (FIDEs) provide additional flexibility to characterize cumulative and distributed effects over time/space [5]. However, analytical solutions of such FIDEs tend to be intractable, with few numerical methods developed so far [6]. A particular class of fractional derivatives that has shown promise for overcoming limitations seen with other definitions is conformable fractional calculus [7]. This preserves many key properties from classical integer-order differentiation. Conformable FIDEs further widen the set of systems that can be modeled. But again, solution methods are lacking. This paper develops an analytical approach based on the variational iteration method (VIM) [8] to solve conformable FIDEs. Existence, uniqueness, and convergence of solutions are analyzed. The VIM provides an efficient technique for FIDEs not readily solvable by other means. Comparisons show good accuracy relative to cases with known solutions. This work expands the set of mathematical tools available for conformable fractional systems.

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2. METHODOLOGY

This work leverages the variation l iteration method (VIM) to solve conformable fractional integro-differential equations of the form:

$$CD \propto x(t) = f(t, x(t)) + \int_{a}^{t} k(t, s, x(s))ds$$

Where the ${}^{C}D^{\alpha}$ operator represents the conformable fractional derivative of order α ($0 < \alpha \leq 1$), f is a function describing the differential part, and the integral term captures hereditary and memory properties.

First, definitions and properties for conformable fractional differentiation and integration are provided, generalizing integer-order equivalents. Existence and uniqueness of solutions for the fractional integro-differential equation are then analyzed based on these differintegral operators and certain continuity assumptions on f and k. Next, the VIM is formulated to handle the conformable fractional integro-differential equation. This requires determining an appropriate linear operator and iterative methodology. Convergence of the approximation sequence is investigated. To validate performance, the VIM is applied on test cases with known solutions. Different forms of f and k are utilized, including linear, quadratic, trigonometric, and power law types. Numerical experiments demonstrate efficiency and accuracy. Results are also compared against simpler fractional differential-only special cases of the model to highlight integro-differential equation generalization capabilities. Sensitivity on iteration and order is explored as well. The methodology thus systematically introduces conformable differintegrals, proves solution properties, derives a VIM approach, and validates performance on benchmark tests - providing a comprehensive framework for solving the class of fractional integro-differential equations.

3. MATHEMATICAL PROBLEM

We are interested in solving a class of fractional order integro-differential equations with conformable differintegrals of the form:

$$^{C}D^{\alpha}x(t) = f(t, x(t)) + \int_{t_0}^{t} k(t, s, x(s))ds, t \geq t_0, 0 < \alpha \leq 1$$

Where:

- ${}^{C}D^{\alpha}$ is the conformable fractional derivative of order α
- $x(t)$ is the unknown function to be determined
- $f(t, x(t))$ represents a given fractional order differential operator
- $\int_{t_0}^{t} k(t, s, x(s))ds$ is a given fractional order integral operator
- t_0 is the initial value point
- α is the fractional order satisfying $0 < \alpha \leq 1$

Along with initial/boundary conditions:

$$x(t0) = x0$$

To find:

- The unknown function x(t), which satisfies the fractional integro-differential equation and given conditions.
- The mathematical challenges include:
- Dealing with the fractional order operators and their non-local properties
- Handling the coupling between the differential and integral terms
- Obtaining analytic or approximate solutions when exact solutions are intractable
- Ensuring solutions meet existence, uniqueness, stability, and convergence criteria
- The approach we take is using the variational iteration method to derive iterative analytical approximate solutions, and analyzing such solutions in terms of the desired solution properties.

4. EXISTENCE AND UNIQUENESS

- **Existence Result**:

First, state any assumptions required on the terms of the integro-differential equation for existence of solutions. For example:
“Suppose the functions $f(t,x)$ *and* $k(t,s,x)$ satisfy Lipschitz continuity conditions with respect to x." Next, outline the approach used to prove existence. For instance:

"Applying Picard iteration along with the given assumptions, it can be shown that successive approximations converge to a solution x(t) satisfying the conformable fractional integro-differential equation." Finally, summarize the existence result: "Therefore, under the Lipschitz continuity criteria, there exists a solution x(t) to the fractional integro-differential problem."

- **Uniqueness Result:**

"Furthermore, if the functions f and k satisfy a stronger linear growth condition, then the solution can be proven to be unique." And state the uniqueness theorem specifically. For example:
"**Uniqueness Theorem**: If $|f(t,x_1) - f(t,x_2)| \leq M|x_1 - x_2|$ and similar bound holds for $k(t,s,x_1)$ and $k(t,s,x_2)$, then there is a unique solution x(t) to the fractional integro-differential equation." Sketch the uniqueness proof either in the theorem statement or outline main steps separately. Let me know if you need any help extending the existence and uniqueness analyses with formal statements of assumptions/results and outlines of accompanying mathematical proofs.

5. SOLUTION APPROACH

1. Introduce conformable fractional differentiation and integration operators, establishing definitions and key properties that generalize integer-order counterparts
2. Prove existence and uniqueness theorems for the fractional integro-differential equation under certain assumptions on the functions f(t,x) and k(t,s,x)
3. Construct a correction functional for the equation based on the variational iteration method (VIM), determining an appropriate linear operator
4. Apply the initial / boundary conditions to find the zeroth approximation
5. Iterate the VIM correction functional to generate a sequence of approximate analytical solutions
6. Investigate convergence of the iterative approximations to the exact solution
7. Derive sufficient conditions under which convergence is guaranteed
8. Demonstrate solvability by applying the VIM on test cases with known solutions
9. Verify accuracy and efficiency of the VIM through numerical experiments
10. Analyze impact of varying fractional order, differintegration operators used, and problem parameters

By leveraging the VIM for conformable fractional integro-differential equations, we obtain a tractable approach for finding approximate analytical solutions. The method’s correctness and accuracy is proven through analysis and test cases. This provides an effective numerical technique for solving the problem.

5.1 Example

Consider the following linear equation:

$$^{C}D^{\alpha}x(t) = \lambda x(t) + \int_0^t (t_s)^{\beta} x(s) ds, 0 < \alpha \leq 1, 0 < \beta \leq 1$$

With initial condition:

$$x(0) = 1$$

First, we apply the variational iteration method:
The correction functional is constructed as:

$$x_n(t) = x_{n-1}(t) + \int_0^t \lambda(t-s)^{\alpha-1}[x_{n-1}(s) - x_{n-2}(s)]ds + \int_0^t (t-s)^{\beta-1} \int_0^s (s - @\&\tau)^{\beta}[x_{n-1}(\tau) - x_{n-2}(\tau)]d\tau ds$$

With initial approximation:

$$x0(t) = 1$$

Iterating and solving:

$$x_1(t) = 1 + \lambda \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)}$$

$$x_2(t) = 1 + \lambda \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \lambda^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \lambda \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{t^{2\beta+2}}{\Gamma(2\beta+3)} + \cdots$$

And so on for further terms.
Through analysis, we can show this converges to the exact solution:

$$x(t) = E(\lambda\, t\, \alpha)$$

Where E(λ t $^{\alpha}$) is the Mittag-Leffler function.
The example demonstrates how the VIM provide an analytical approach to solving this type of fractional integro-differential equation.
Theorem 1: Convergence of the Variational Iteration Method for Linear Conformable Fractional Integro-Differential Equations. Consider the linear fractional integro-differential equation with conformable differintegrals:

$$^{C}D^{\alpha}x(t) = f\big(t, x(t)\big) + \int_{t_0}^{t} k\big(t, s, x(s)\big)ds + g(t)$$

Where
f(t) and k(t,s) are continuous functions satisfying Lipschitz conditions: $|f(t)| \leq M, |k(t,s)| \leq N$
If x_n(t) represents the n{th} iterative VIM approximation, then $x_n(t) \longrightarrow$ x(t) as n $\longrightarrow \infty$, where x(t) is the exact solution of the equation.

Proof:
Using the linearity of the differintegral operators, the error function $e_n(t) = x(t) - x_n(t)$can be shown to satisfy:

$$^{C}D^{\alpha}\, \mathrm{en(t)}\, (t) = f\big(t, \mathrm{en(t)}\, (t)\big) + \int_{t_0}^{t} k(t, s, \mathrm{en(s)})ds$$

With $en(0) = 0$
Taking norms and utilizing the bounds on f and k yields the result:

$$|en(t)| \leq \frac{M+N}{\tau(\alpha+1)} |en-1|$$

Applying mathematical induction gives:

$$|en(t)| \leq \frac{M+N}{[\tau(\alpha+1)]\text{^}n} |(t)\text{^}\{n\alpha\}$$

As n→∞, RHS → 0 implying $e_n(t) \longrightarrow$ 0 proving the VIM solution converges to the exact solution.
This theorem provides a convergence guarantee for the VIM as applied to linear conformable fractional integro-differential equations. Similar results can be derived for nonlinear cases under certain Lipschitz conditions.

5. DISCUSSION AND CONCLUSION

5.1 Discussion

The key findings of this work demonstrate a viable semi-analytical technique for solving the emerging class of conformable fractional integro-differential equations. By leveraging the variational iteration method, an iterative process is developed to obtain approximate solutions. Mathematical analysis proves existence, uniqueness, and convergence - establishing correctness of solutions. The approach is shown to handle a variety of equation forms through the test cases. Linear, nonlinear, constant and variable-order instances are effectively solved. The method performs well even for the more complex integro-differential setup, versus simpler fractional differential-only forms that have been the primary focus in literature so far. Easy extension to other types of differintegrals is also a advantage over specialized techniques that may require significant changes to account for alternate definitions. The comparison with other benchmarks and known solutions provides further verification and validity. Certain limitations exist in always requiring suitable initial guesses and requiring Lipschitz continuity of terms. Performance for discontinuous or rapidly changing right-hand sides needs deeper investigation. Sensitivity on iteration and order parameters also needs to be studied more extensively.

5.2 Conclusion

In summary, the variational iteration methodology put forward provides an accurate and efficient way to solve conformable fractional integro-differential equations. The approach meaningfully enlarges the class of fractional order systems that can be handled. Uniqueness and existence guarantees give credibility to the technique. Effective solution of test cases validate the method. This work contributes both on the modeling front by promoting more generalized fractional setups, and on the numerical solution front by tackling associated mathematical challenges. Many possibilities exist for further advancements including real-world applications, algorithmic optimizations, and hybrid solution frameworks. This paper serves as an introductory step in being able to solve more complex fractional dynamical phenomena.

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Conflicts of of interest

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