Babylonian Journal of Mathematics
Vol. **2024**, **pp**. 1–12
DOI:https://doi.org/10.58496/BJM/2024/001; ISSN:3006-113X
https://mesopotamian.press/journals/index.php/mathematics

Research Article

Substitutional Based Gauss-Seidel Method for Solving System of Linear Algebraic Equations

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ARTICLE INFO

Article History

Received 09 Nov 2023
Revised 02 Dec 2023
Accepted 15 Dec 2023
Published 10 Jan 2024

Keywords

Differential –difference algebra

Time delay

Discrete time delay

continuously distributed delays

ABSTRACT

In this research paper a new modification of Gauss-Seidel method has been presented for solving the system of linear algebraic equations. The systems of linear algebraic equations have an important role in the field of science and engineering. This modification has been developed by using the procedure of Gauss-Seidel method and the concept of substitution techniques. Developed modification of Gauss-Seidel method is a fast convergent as compared to Gauss Jacobi's method, Gauss-Seidel method and successive over-relaxation (SOR) method. It is applicable to both diagonally dominant and positive definite symmetric systems of linear algebraic equations. Its solution has been compared with the Gauss Jacobi's method, Gauss-Seidel method and Successive Over-Relaxation method by taking different systems of linear algebraic equations and found that, it was reducing to the number of iterations and errors in each problem.

1. INTRODUCTION

Consider the general form of the system of linear algebraic equations as

$$AX = b \tag{1}$$

where $A = [a_{i,j}]$ be a non-singular square matrix, $X = [x_i]$ *and* $b = [b_i]$, $i, j = 1, 2, 3, \ldots, n$ [1-2]. Further, this system of linear algebraic equations has categorized in two ways; homogeneous and non-homogeneous. If $b = 0$ in eq. (1), then it is called a homogeneous system of linear equations otherwise non-homogeneous system of linear equations. The concept of system of linear equations was introduced by the famous mathematician Rene Descartes in the Europe in 1637 [3-5].

The system of linear algebraic equations has an important role for solving different physical problems of engineering and science. It is widely used in mechanical systems, electrical circuits, transportation problems operational research, physics, engineering, statistics and social sciences [6-9]. The physical problems in the above fieldshave been solved by using direct and indirect methods [7]. Among the direct methods, the Cramer's rule, Gauss Elimination method, Gauss Jordan method, and LU Decomposition method are well-known direct methods for solving the system of linear algebraic equations [4-5], [10-11]; whereas, the Gauss Jacobi's method, Gauss Seidel method and successive over relaxation method are best indirect methods for solving a large system of linear algebraic equations [3],[12-14]. Actually, the indirect methods are iterative methods which provide approximate solutions; these methods are powerful tools for solving a large system through the computer programming. Large systems of linear equations can't be easily solved by using direct methods because they required more time and a lot of efforts [15], that is why, indirect methods mean iterative methods are given more preference for solving large systems of linear equations with the help of computer programming in a short time. There are not enough

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numerical methods for solving every system of linear algebraic equations because the developed methods have some restrictions or specification for using and cannot be apply for every system of linear equations. The system of linear equations has different types such as diagonally dominant, strictly diagonally dominant, ill-condition, well-conditioned, consistent, inconsistent etc. [16]. The system of linear equations which have solution either unique or infinitely many are known as consistent whereas the system which has no solution are known as inconsistent system of linear equations [3]. The Gauss Jacobi's method and Gauss Seidel method are only applicable for solving diagonally dominant system of linear equations; it means every iterative method is used for solving a specific type of system of linear equations. That is why, different researchers and scholars have developed various numerical methods for solving system of linear equations and they are also trying to develop new efficient and fast convergent methods. Previous developed iterative methods and their working rules are discussed here.

1.1. Gauss Jacobi's iterative method

Gauss Jacobi's method is an old iterative method which is used for solving the diagonally dominant system of linear algebraic equations [3], [12]. The working rule for Gauss Jacobi's method is explained here by taking a 3×3 system of linear algebraic equations satisfying the conditions of diagonally dominant.

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \qquad (2)$$

Now, rearrange the system for $x, y \; and \; z$.

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \\ y &= \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \\ z &= \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \end{aligned}\right\}$$

The Gauss Jacobi's method is an iterative method so; write the above formula in an iterative form as

$$\left.\begin{aligned} x^{n+1} &= \frac{1}{a_{11}}(b_1 - a_{12}y^n - a_{13}z^n) \\ y^{n+1} &= \frac{1}{a_{22}}(b_2 - a_{21}x^n - a_{23}z^n) \\ z^{n+1} &= \frac{1}{a_{33}}(b_3 - a_{31}x^n - a_{32}y^n) \end{aligned}\right\} \qquad (3)$$

This is an iterative formula Gauss Jacobi's method. Here $n = 0, 1, 2, 3, \ldots$.

For solving the given system of linear algebraic equations, take initial guesses as $(x^0, y^0, z^0) = (k, k, k)$(where k is a constant) for getting first approximate solution and again, use the solution set of first iteration as initial guesses for the second approximate solution and so on.

1.2. Gauss-Seidel iterative method

Gauss Seidel method is the modification of Gauss Jacobi's method [3], [12]. It uses the updated previous calculated value for the second value in the every iteration. This method also works for the diagonally dominant system of linear algebraic equations. The procedure of Gauss Seidel method is also explained here by taking a 3×3 system of linear algebraic equations given by eq. (2) which is

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\}$$

Now, rearrange the system of linear for $x, y \; and \; z$.

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \\ y &= \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \\ z &= \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \end{aligned}\right\}$$

Gauss Seidel method uses initial guesses and updated calculated value and its iterative formula is defined as

$$\left.\begin{aligned} x^{n+1} &= \frac{1}{a_{11}}(b_1 - a_{12}y^n - a_{13}z^n) \\ y^{n+1} &= \frac{1}{a_{22}}(b_2 - a_{21}x^{n+1} - a_{23}z^n) \\ z^{n+1} &= \frac{1}{a_{33}}(b_3 - a_{31}x^{n+1} - a_{32}y^{n+1}) \end{aligned}\right\} \tag{4}$$

This is an iterative formula for the Gauss Seidel method. The iterative formula is used for finding the solution of given system of linear equation by assuming an initial guesses as $(x^0, y^0, z^0) = (k, k, k)$ where k is a constant and updated calculated value for the solution in the first approximation. Similarly, assume the solution of first approximation as initial guesses and proceed for the solution in the second approximation as above and so on.

1.3. Successive Over-Relaxation method (SOR method)

Successive over-relaxation method is an iterative method and a variant of Gauss Seidel method [12-13]. This method accelerates the Gauss Seidel method by using relaxation factor ω. The iterative formula for SOR method is defined as

$$x_i^{p+1} = (1-\omega)x_i^p + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{p+1} - \sum_{j>i} a_{ij}x_j^p\right), \quad i = 1,2,3, \dots, n \tag{5}$$

In this method if $\omega = 1$, then it simplifies the Gauss Seidel method and if $0 < \omega < 1$ then it is called Successive under-relaxation method and this method is convergent if $\omega \in (0,\ 2)$ [17].

2. RESEARCH METHODOLOGY

The algorithm of new developed method has been generated by using the concept of substitution method and the procedure of Gauss-Seidel method. For the derivation, consider a 2×2 system of non-homogeneous linear equations in the form

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \tag{6}$$

where$a_1, a_2, b_1, b_2, c_1, c_2$ are constants and x and y are variables. From 2nd equation of eq. (6), the value of y is obtained and then it is substituted in the first equation of eq. (6) and rearranged for x in the form

$$x = \frac{b_2}{a_1b_2 - b_1a_2}\left[c_1 - \frac{b_1c_2}{b_2}\right] \tag{7}$$

Again, from the first equation of eq. (6), the value of x is obtained and then substituted in the second equation of eq. (6) and rearranged for y in the form

$$y = \frac{a_1}{a_1b_2 - b_1a_2}\left[c_2 - \frac{c_1a_2}{a_1}\right]. \tag{8}$$

Here eq. (6) and eq. (8) represent the exact solution of a given 2×2system of linear equations. Similarly, consider a 3×3 system of non-homogeneous linear algebraic equations in the form

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}. \tag{9}$$

From third equation of eq. (9), obtain the value of z.

$$z = \frac{d_3}{c_3} - \frac{a_3}{c_3}x - \frac{b_3}{c_3}y. \tag{10}$$

From second equation of eq. (9), obtain the value of y.

(11)

$$y = \frac{d_2}{b_2} - \frac{a_2}{b_2}x - \frac{c_2}{b_2}z$$

From first equation of eq. (9), obtain the value of x.

$$x = \frac{d_1}{a_1} - \frac{b_1}{a_1}y - \frac{c_1}{a_1}z \qquad (12)$$

Now, substitute eq. (10) and eq. (11) in the first equation of eq. (9). After substituting and rearranging it provides

$$x = \frac{b_2c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_1b_2a_3)}\left[d_1 - \frac{b_1d_2}{b_2} - \frac{c_1d_3}{c_3} + \frac{c_1b_3}{c_3}y + \frac{b_1c_2}{b_2}z\right] \qquad (13)$$

Similarly, substitute eq. (10) and eq. (12) in the second equation of eq. (9) and rearranging. After rearranging it becomes

$$y = \frac{a_1c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_2b_3a_1)}\left[d_2 - \frac{d_1a_2}{a_1} - \frac{c_2d_3}{c_3} + \frac{c_2a_3}{c_3}x + \frac{c_1a_2}{a_1}z\right]. \qquad (14)$$

Now, substitute eq. (11) and eq. (12) in the third equation of eq. (9) and rearranging it. After rearranging, it becomes

$$z = \frac{a_1b_2}{(a_1b_2c_3 - b_2a_3c_1 - c_2b_3a_1)}\left[d_3 - \frac{d_1a_3}{a_1} - \frac{b_3d_2}{b_2} + \frac{b_3a_2}{b_2}x + \frac{b_1a_3}{a_1}y\right].$$

Finally, eq. (13), eq. (14) and eq. (15) can be written in an iterative formula as (15)

$$\left.\begin{aligned} x^{i+1} &= \frac{b_2c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_1b_2a_3)}\left[d_1 - \frac{b_1d_2}{b_2} - \frac{c_1d_3}{c_3} + \frac{c_1b_3}{c_3}y^i + \frac{b_1c_2}{b_2}z^i\right] \\ y^{i+1} &= \frac{a_1c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_2b_3a_1)}\left[d_2 - \frac{d_1a_2}{a_1} - \frac{c_2d_3}{c_3} + \frac{c_2a_3}{c_3}x^{i+1} + \frac{c_1a_2}{a_1}z^i\right] \\ z^{i+1} &= \frac{a_1b_2}{(a_1b_2c_3 - b_2a_3c_1 - c_2b_3a_1)}\left[d_3 - \frac{d_1a_3}{a_1} - \frac{b_3d_2}{b_2} + \frac{b_3a_2}{b_2}x^{i+1} + \frac{b_1a_3}{a_1}y^{i+1}\right] \end{aligned}\right\} \qquad (16)$$

where$i = 0, 1, 2, 3, \ldots$.

Eq. (16) is a developed an iterative formula for finding the solution of eq. (9). Using same techniques, it can be expanded for a$n \times n$ system of non-homogeneous linear algebraic equations.

Initially, developed iterative formula requires an initial guess i.e. $(x^0, y^0, z^0) = (k, k, k)$where k is any constant.

Now, for the first iteration, put $i = 0$ in eq. (16) and use$(x^0, y^0, z^0) = (k, k, k)$. Then

$$x^1 = \frac{b_2c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_1b_2a_3)}\left[d_1 - \frac{b_1d_2}{b_2} - \frac{c_1d_3}{c_3} + \frac{c_1b_3}{c_3}(\boldsymbol{k}) + \frac{b_1c_2}{b_2}(\boldsymbol{k})\right] = m_1y^1$$

$$= \frac{a_1c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_2b_3a_1)}\left[d_2 - \frac{d_1a_2}{a_1} - \frac{c_2d_3}{c_3} + \frac{c_2a_3}{c_3}(\boldsymbol{m_1}) + \frac{c_1a_2}{a_1}(\boldsymbol{k})\right] = l_1$$

$$z^1 = \frac{a_1b_2}{(a_1b_2c_3 - b_2a_3c_1 - c_2b_3a_1)}\left[d_3 - \frac{d_1a_3}{a_1} - \frac{b_3d_2}{b_2} + \frac{b_3a_2}{b_2}(\boldsymbol{m_1}) + \frac{b_1a_3}{a_1}(\boldsymbol{l_1})\right] = q_1$$

Similarly, for the second iteration, put $i = 1$ in eq. (16) and use initial guesses as $(x^1, y^1, z^1) = (m_1, l_1, q_1)$ then

$$x^2 = \frac{b_2c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_1b_2a_3)}\left[d_1 - \frac{b_1d_2}{b_2} - \frac{c_1d_3}{c_3} + \frac{c_1b_3}{c_3}(\boldsymbol{l_1}) + \frac{b_1c_2}{b_2}(\boldsymbol{q_1})\right] = m_2$$

$$y^2 = \frac{a_1c_3}{(a_1b_2c_3 - b_1a_2c_3 - c_2b_3a_1)}\left[d_2 - \frac{d_1a_2}{a_1} - \frac{c_2d_3}{c_3} + \frac{c_2a_3}{c_3}(\boldsymbol{m_2}) + \frac{c_1a_2}{a_1}(\boldsymbol{q_1})\right] = l_2$$

$$z^2 = \frac{a_1b_2}{(a_1b_2c_3 - b_2a_3c_1 - c_2b_3a_1)}\left[d_3 - \frac{d_1a_3}{a_1} - \frac{b_3d_2}{b_2} + \frac{b_3a_2}{b_2}(\boldsymbol{m_2}) + \frac{b_1a_3}{a_1}(\boldsymbol{l_2})\right] = q_2$$

and continue same procedure up to required accuracy.

3. CONVERGENCE CRITERIA

Developed modified algorithm is valid for a non-homogeneous diagonally-dominant system of linear algebraic equations and for a positive definite, symmetric coefficient matrix.. It is fast convergent method as compared toGauss Jacobi's method, Gauss-Seidel method and SOR method.

Definition: A system of non-homogeneous linear equations is said to be in diagonally dominant [12], [18] if it satisfies the condition

$$\left.\begin{aligned} |a_{11}| &\geq |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\ |a_{22}| &\geq |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\ |a_{33}| &\geq |a_{31}| + |a_{32}| + \cdots + |a_{3n}| \\ &\vdots \qquad\quad \vdots \qquad\qquad \vdots \\ |a_{nn}| &\geq |a_{n1}| + |a_{n2}| + \cdots + |a_{n\,n-1}| \end{aligned}\right\} \qquad (17)$$

where a_{ij},i, j = 1, 2, 3… n, are coefficients of variables used in the system of linear equations.

4. RESULTS AND DISCUSSION

In this section, results of proposed method are compared with the Gauss Jacobi's method, Gauss Seidel method and Successive over-relaxation method by using Dev C++ programing.

Example1: Solve the system of linear equations [6], [8], [12]:

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Solution:

TABLE I (a). Solution of Example 1

Iterative Methods	Approximate Solution	Exact solution	No of iterations	Absolute Errors	Required Accuracy%
Gauss Jacobi's method	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3;$ $x_4 = 0$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	22	0,0,0,0	0.0000000001
Gauss Seidel method	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3;$ $x_4 = 0$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	12	0, 0, 0, 0,	0.0000000001
Proposed method	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3x_4 = 2.88353e^{-020}$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	6	0,0, 0,0	0.0000000001

In the table 1(a), the numerical solutions of system of linear algebraic equations using Gauss Jacobi's method, Gauss Seidel method and proposed method have been presented. The Gauss Jacobi's method, Gauss Seidel method and proposed method show the almost same solution as exact solution within 22, 12, 6 number of iterations respectively. In the above table 1(a); the solutions are obtained at 0.0000000001 percentage accuracy with the initial guess as zeros (means all values of x are taken as zero). From the above table; it is clear that the proposed method performed almost same solution as exact solution within less number of iterations as compared to Gauss Seidel method and Gauss Jacobi's method. The absolute errors of last iteration of each method are mentioned in the table and these results have been obtained by using Dev C++ software.

TABLE I (b). SOLUTION OF EXAMPLE 1

Relaxation factor ω	Approximate Solution of SOR METHOD	Exact solution	No of iterations	Absolute Errors	Required Accuracy%
$\boldsymbol{\omega = 1.05}$	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3;$ $x_4 = 7.21904e^{-015}$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	12	Error $x_1 = 0$ Error $x_2 = 0$ Error $x_3 = 0$ Error$x_4 = 1.516e^{-011}$	0.0000000001
$\boldsymbol{\omega = 1.15}$	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3;$ $x_4 = 1.09646e^{-013}$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	17	Error $x_1 = 0$ Error $x_2 = 0$ Error $x_3 = 0$ Error$x_4 = 8.40623e^{-011}$	0.0000000001

$\boldsymbol{\omega = 1.25}$	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3$ $x_4 = 1.03417e^{-013}$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	22	Error $x_1 = 0$ Error $x_2 = 0$ Error $x_3 = 0$ Error$x_4 = 5.17086e^{-011}$	0.0000000001
$\boldsymbol{\omega = 1.5}$	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3$ $x_4 = 3.27366e^{-013}$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	44	Error $x_1 = 0$ Error $x_2 = 0$ Error $x_3 = 0$ Error$x_4 = 9.82997e^{-011}$	0.0000000001
$\boldsymbol{\omega = 1.75}$	$x_1 = 1;$ $x_2 = 2;$ $x_3 = 3$ $x_4 = -4.07241e^{-013}$	$x_1 = 1; x_2 = 2; x_3 = 3; x_4 = 0$	110	Error $x_1 = 0$ Error $x_2 = 0$ Error $x_3 = 0$ Error$x_4 = 9.50229e^{-011}$	0.0000000001

Table 1(b) represents the numerical analysis of the SOR method of example 1. In this table, the numerical solutions of SOR method are obtained by choosing different values of relaxation factor in the iterative formula of SOR method and found that SOR method taking same number of iterations as taken by Gauss Seidel method; which is shown in the table 1(a), whereas; it takes more iterations as compared to proposed method for obtaining solution. The solutions of SOR method calculated at $\omega = 1.05,\ \omega = 1.15, \omega = 1.25, \omega =\ 1.5\ and \omega = 1.75$ with the initial guesses as zeros (means all variables initially considered as zeros) and required accuracy percentage as 0.0000000001 represent in the table 1(b) have been compared with the solution of proposed method represent in the table 1(a) and found that the proposed method shows less number of iterations and more accuracy than the all SOR solutions represent in the table 1(b). The second and fifth columns of table 1(b) show the solutions of SOR method and absolute errors respectively obtained in the number of iterations written in the fourth column of table 1(b).

Example 2: Solve the system of linear equations[9], [19]:

$$4x_1 - x_2 - x_4 = 0$$
$$-x_1 + 4x_2 - x_3 - x_5 = 5$$
$$-x_2 + 4x_3 - x_6 = 0$$
$$-x_1 + 4x_4 - x_5 = 6$$
$$-x_2 - x_4 + 4x_5 - x_6 = -2$$
$$-x_3 - x_5 + 4x_6 = 6$$

Solution:

TABLE 2 (a). SOLUTION OF EXAMPLE 2

Iterative Methods	Approximate Solution	Exact solution	No of iterations	Absolute Errors	Required Accuracy%
Gauss Jacobi's method	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	32	Error $x_1 = 1.19209e^{-005}$ Error $x_2 = 0$ Error $x_3 = 1.19209e^{-005}$ Error $x_4 = 0$ Error $x_5 = 2.38419e^{-005}$ Error $x_6 = 0$	0.00001
Gauss Seidel method	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	18	Error $x_1 = 5.96046e^{-006}$ Error $x_2 = 1.19209e^{-005}$ Error $x_3 = 5.96046e^{-006}$ Error $x_4 = 1.19209e^{-005}$ Error $x_5 = 1.19209e^{-005}$ Error $x_6 = 0$	0.00001
Proposed method	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	8	Error $x_1 = 5.96046e^{-006}$ Error $x_2 = 1.19209e^{-005}$ Error $x_3 = 0$ Error $x_4 = 0$ Error $x_5 = 0$ Error $x_6 = 0$	0.00001

The numerical solutions of Gauss Jacobi's method, Gauss Seidel method and proposed method have been presented in the table 2(a). All these three solutions sets are obtained at 0.00001 percentage accuracy with the initial guesses as zeros (means all values of x are taken as zero initially). The Gauss Jacobi's method, Gauss Seidel method and proposed method showthe same solution sets as exact solution set within the 32, 18 and 8 numbers of iterations respectively. The proposed method reduces to the number of iterations and produce quickly solutions as compared to Gauss Jacobi's method and Gauss Seidel method. In this table, the errors column shows the absolute errors of the solution set in the last iteration of corresponding methods. These solution sets have been obtained by using Dev C++ software.

The numerical solution of SOR method at different values of relaxation factor, exact solution, and numbers of iterations carried out by SOR method, corresponding absolute errors of last iteration and required accuracy percentage have been presented in the Table 2(b). From the table 2(b) and table 2(c); it is observed that the SOR method takes more number of iterations for getting solution as the relaxation factor ω approaches to 2 and the SOR method uses more number of iterations as the relaxation factor ω approaches to 1 from the relaxation factor $\omega = 1.13$. It provides required solution with least number of iterations using the value of relaxation factor $\omega = 1.13$ and 1.12. These all solution sets present in table 2(b) and calculated for table 2(c) at different value of ω take more iteration as compared to the proposed method which is present in the table 2(a).

TABLE 2 (b). Solution of Example 2

Relaxation factor $\boldsymbol{\omega}$	Approximate Solution of SOR method	Exact solution	No of iterations	Absolute Errors	Required Accuracy%
$\boldsymbol{\omega = 1.10}$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	12	Error $x_1 = 1.19209e^{-005}$ Error $x_2 = 1.19209e^{-005}$ Error $x_3 = 5.96046e^{-006}$ Error $x_4 = 1.19209e^{-005}$ Error x_5= $5.96046e^{-006}$ Error x_6= 0	0.00001
$\boldsymbol{\omega = 1.15}$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	10	Error $x_1 = 6.55651e^{-005}$ Error $x_2 = 4.76837e^{-005}$ Error $x_3 = 1.19209e^{-005}$ Error $x_4 = 0$ Error x_5= $1.19209e^{-005}$ Error x_6= 0	0.00001
$\boldsymbol{\omega = 1.25}$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	13	Error $x_1 = 5.96046e^{-005}$ Error $x_2 = 4.76837e^{-005}$ Error $x_3 = 2.98023e^{-005}$ Error x_4= 0 Error $x_5 = 1.19209e^{-005}$ Error x_6= 0	0.00001
$\boldsymbol{\omega = 1.5}$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	26	Error $x_1 = 0$ Error $x_2 = 1.19209e^{-005}$ Error $x_3 = 1.19209e^{-005}$ Error x_4= 0 Error $x_5 = 5.96046e^{-006}$ Error $x_6 = 0$	0.00001
$\boldsymbol{\omega = 1.75}$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	$x_1 = 1$ $x_2 = 2$ $x_3 = 1$ $x_4 = 2$ $x_5 = 1$ $x_6 = 2$	62	Error $x_1 = 0$ Error $x_2 = 1.19209e^{-005}$ Error $x_3 = 5.96046e^{-006}$ Error x_4= 0 Error $x_5 = 1.19209e^{-005}$ Error $x_6 = 0$	0.00001

TABLE 2 (c). Solution of Example 2 Using Sor Method

Value of Relaxation factor $\boldsymbol{\omega}$	1.05	1.10	1.11	1.12	1.13	1.35	1.14	1.15	1.25	1.5	1.75

No. of iterations	**15**	**12**	**11**	**9**	**9**	**10**	**10**	**10**	**13**	**26**	**62**

Example3: Solve the system of linear equations [20]:

$$10x_1 - 8x_2 = -6$$
$$-8x_1 + 10x_2 - x_3 = 9$$
$$-x_2 + 10x_3 = 28$$

Solution:

TABLE 3 (a). Solution of Example 3

Iterative Methods	Approximate Solution	Exact solution	No of iterations	Absolute Errors	Required Accuracy%
Gauss Jacobi's method	$x_1 = 0.999995$ $x_2 = 2$ $x_3 = 3$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	59	Error $x_1 = 0.000101328$ Error $x_2 = 0.000345707$ Error $x_3 = 0$	0.0000000001
Gauss Seidel method	$x_1 = 0.999995$ $x_2 = 2$ $x_3 = 3$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	31	Error $x_1 = 0.0002563$ Error $x_2 = 0.000214577$ Error $x_3 = 0$	0.0000000001
Proposed method	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	6	Error $x_1 = 5.96046e^{-006}$ Error $x_2 = 0$ Error $x_3 = 0$	0.0000000001

The numerical solutions, no. of iterations and errors are obtained by using Gauss Jacobi's method; Gauss Seidel method and proposed method have been presented in the table 3(a). The Gauss Jacobi's method and Gauss Seidel method almost shows the same solution set as exact solution within the 59 and 31 numbers of iterations respectively whereas, the proposed method shows same solution set as exact solution within only 6 numbers of iterations. In this problem all solution sets are obtained with 0.0000000001 percentage accuracy and the initial guesses used for each method are zeros (it means all value of x are zero initially). Fifth column of the table 3(a) shows the absolute errors of each method obtained in the last iteration. Overall, it is observed from the table 3(a); the proposed method shows less number of iterations and performed very quickly solutions as compared to Gauss Jacobi's method and Gauss Seidel methods.

TABLE 3 (b). Solution of Example 2

Relaxation factor $\boldsymbol{\omega}$	Approximate Solution of SOR method	Exact solution	No of iterations	Absolute Errors	Required Accuracy%
$\boldsymbol{\omega = 1.10}$	$x_1 = 0.999997$ $x_2 = 2$ $x_3 = 3$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	25	Error $x_1 = 0.000214577$ Error $x_2 = 0.000166893$ Error $x_3 = 0$	0.0000000001
$\boldsymbol{\omega = 1.15}$	$x_1 = 0.999998$ $x_2 = 2$ $x_3 = 3$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	22	Error $x_1 = 0.000196695$ Error $x_2 = 0.000143051$ Error $x_3 = 0$	0.0000000001
$\boldsymbol{\omega = 1.25}$	$x_1 = 0.999999$ $x_2 = 2$ $x_3 = 3$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	15	Error $x_1 = 0.000149012$ Error $x_2 = 8.34465e^{-005}$ Error $x_3 = 0$	0.0000000001
$\boldsymbol{\omega = 1.5}$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	$x_1 = 1$ $x_2 = 2$ $x_3 = 3$	26	Error $x_1 = 5.96046e^{-006}$ Error $x_2 = 0$ Error $x_3 = 0$	0.0000000001

TABLE 3 (c). Solution of Example #3 Using Sor Method

Value of Relaxation factor ω	1.05	1.10	1.15	1.20	1.23	1.24		1.25	1.26	1.27	1.30	1.5
No. of iterations	**29**	**25**	**22**	**20**	**17**	**16**		**15**	**14**	**15**	**16**	**26**

Table 3(b) and table 3(c) represent the complete analysis of SOR method. All solution sets of examples 3; obtained by SOR method with corresponding absolute errors, number of iterations and relaxation factors have been presented in the table 3(b). These all solution sets are very close to exact solution of example 3 but, the appropriate solution set is obtained by choosing a suitable value of relaxation factor that is $\omega = 1.25$ as shown in the table 3(b); which also minimizes the error and takes a smaller number of iterations. So, at $\omega = 1.25$ the SOR method produce a good solution of example 3. For more accurate solution the analysis has carried out by taking different value of relaxation factor in the SOR method as shown in the table 3(c) and found that $\omega = 1.26$is abest appropriate relaxation factor for finding the best solution using SOR method. Finally, the best obtained solution set has been compared with the solution set obtained by proposed method and observe that the proposed method uses only 6 numbers of iterations for same solution set with minimum absolute errors whereas the SOR method uses minimum 14 numbers of iterations for the same solution. Hence, from the table 3(a), table 3(b) and table 3(c); it is clear that the proposed method is a best method for solving system of linear algebraic solutions. In the table 3(b) and table 3(c), the initial guesses and required accuracy percentage are same as table 3(a).

Example4: Solve the system of linear equations [20]:

$$4x_1 - x_2 = 1$$
$$-x_1 + 4x_2 - x_3 = 2$$
$$-x_2 + 4x_3 - x_4 = 2$$
$$-x_3 + 4x_4 - x_5 = 2$$
$$-2x_4 + 4x_5 = 2$$

Solution: The exact solution set is $(x_1,\ x_2,\ x_3,\ x_4,\ x_5) = (0.464088,\ 0.856354,\ 0.961326,\ 0.98895,\ 0.994475)$

TABLE 4 (a). Solution of Example 4

Iterative Methods	Approximate Solution	No of iterations	Absolute Errors	Required Accuracy%
Gauss Jacobi's method	$x_1 = 0.464088$ $x_2 = 0.856354$ $x_3 = 0.961326$ $x_4 = 0.98895$ $x_5 = 0.994475$	24	Error $x_1 = 1.05669e^{-038}$ Error $x_2 = 0$ Error x_3= 0 Error x_4= 0 Error x_5= 0	0.0000000001
Gauss Seidel method	$x_1 = 0.464088$ $x_2 = 0.856354$ $x_3 = 0.961326$ $x_4 = 0.98895$ $x_5 = 0.994475$	13	Error $x_1 = 9.1894e^{-039}$ Error $x_2 = 5.96046e^{-006}$ Error $x_3 = 5.96046e^{-006}$ Error x_4= 0 Error x_5= 0	0.0000000001
Proposed method	$x_1 = 0.464088$ $x_2 = 0.856354$ $x_3 = 0.961326$ $x_4 = 0.98895$ $x_5 = 0.994475$	5	Error $x_1 = 0.000110269$ Error $x_2 = 0$ Error $x_3 = 2.38419e^{-005}$ Error x_4= 0 Error x_5= 0	0.0000000001

Methods' name, approximate solutions, no. of iterations, absolute errors and required accuracy percentage used in each method have been presented in the table 4(a). For this problem; the Gauss Jacobi's method, Gauss Seidel method and proposed method use 24, 13 and 5 numbers of iterations for obtaining solution sets with 0.0000000001 percentage accuracy. These solution sets almost are same as the exact solution set. In this table, the fifth column shows the absolute errors obtained in the last iteration of each method and these all results have been obtained by using Dev C++ software. The initial guesses have been used for this problem are zeros (means all variables considered as zero initially).

TABLE 4 (b). Solution of Example 4

Relaxation factor ω	Approximate Solution of SOR method	No of iterations	Absolute Errors	Required Accuracy%

$\boldsymbol{\omega = 1.05}$	$x_1 = 0.464088$ $x_2 = 0.856354$, $x_3 = 0.961326$ $x_4 = 0.98895$, $x_5 = 0.994475$	10	Error $x_1 = 1.19209e^{-005}$ Error $x_2 = 5.96046e^{-006}$, Error $x_3 = 0$ Error $x_4 = 0$, Error $x_5 = 0$	0.0000000001
$\boldsymbol{\omega = 1.10}$	$x_1 = 0.464088$ $x_2 = 0.856354$,$x_3 = 0.961326$ $x_4 = 0.98895$,$x_5 = 0.994475$	9	Error $x_1 = 8.9407e^{-006}$, Error $x_2 = 0$ Error $x_3 = 5.96046e^{-006}$ Error $x_4 = 5.96046e^{-006}$, Error $x_5 = 0$	0.0000000001
$\boldsymbol{\omega = 1.15}$	$x_1 = 0.464088$ $x_2 = 0.856354$,$x_3 = 0.961326$ $x_4 = 0.98895$,$x_5 = 0.994475$	10	Error $x_1 = 1.49012e^{-005}$ Error $x_2 = 2.38419e^{-005}$, Error $x_3 = 5.96046e^{-006}$ Error $x_4 = 5.96046e^{-006}$, Error x_5= 0	0.0000000001
$\boldsymbol{\omega = 1.25}$	$x_1 = 0.464088$, $x_2 = 0.856354$ $x_3 = 0.961326$ $x_4 = 0.98895$, $x_5 = 0.994475$	12	Error $x_1 = 2.38419e^{-005}$ Error $x_2 = 9.53674e^{-005}$, Error $x_3 = 2.38419e^{-005}$ Error $x_4 = 2.98023e^{-005}$, Error x_5= 0	0.0000000001
$\boldsymbol{\omega = 1.5}$	$x_1 = 0.464088$ $x_2 = 0.856354$, $x_3 = 0.961326$ $x_4 = 0.98895$, $x_5 = 0.994475$	23	Error $x_1 = 2.68221e^{-005}$ Error $x_2 = 1.19209e^{-005}$, Error $x_3 = 4.76837e^{-005}$ Error $x_4 = 1.19209e^{-005}$, Error x_5= 0	0.0000000001
$\boldsymbol{\omega = 1.75}$	$x_1 = 0.464088$ $x_2 = 0.856354$, $x_3 = 0.961326$ $x_4 = 0.98895$, $x_5 = 0.994475$	59	Error $x_1 = 1.49012e^{-005}$, Error $x_2 = 5.96046e^{-006}$ Error $x_3 = 5.96046e^{-006}$Error $x_4 = 5.96046e^{-006}$ Error x_5= 0	0.0000000001

TABLE 4 (c). SOLUTION OF EXAMPLE #3 USING SOR METHOD

Value of Relaxation factor $\boldsymbol{\omega}$	1.05	1.06	**1.07**	1.08	1.09	1.10	1.13	1.15	1.25	1.5	1.75
No. of iterations	10	9	**8**	8	8	9	10	10	12	23	59

Table 4(b) and table 4(c) show the complete numerical analysis of SOR method. In table 4(b); the values of relaxation factor have been chosen randomly as $\omega = 1.05$, $\omega = 1.10$, $\omega = 1.15$, $\omega = 1.25$, $\omega = 1.5$ and $\omega = 1.75$ and found that the SOR method takes 10, 9, 10, 12, 23 and 59 numbers of iterations respectively for getting the solution set. In table 4(b); all solution sets have been calculated by assuming initial guesses as zeros (it means all variable initially taken as zeros) and required accuracy percentage 0.0000000001. The fourth column of table 4(b) represents the absolute errors calculated in the last iteration by SOR method. From table 4(b), it is noticed that the SOR method provides an accurate solution set at $\omega = 1.10$ but, for getting more accurate solution set, it is further analyzed for the different value of relaxation factor near $\omega = 1.10$ and found that the SOR method performed more accurate solution set at $\omega = 1.07$ with minimum number of iterations and absolute errors. Finally, the more accurate solution set obtained by SOR method has been compared with the solution set of proposed method given in the table 4(a) that the proposed method performed more accurate solution set with less numbers of iterations and absolute errors.

From these four examples; it is clearly observed that the proposed method is fast convergent as compared to Gauss Jacobi's method, Gauss Seidel method and Successive Over-Relaxation (SOR) method.

5. CONCLUSION

In this research paper a new modification of Gauss Seidel method has been presented for solving non-homogeneous linear system of algebraic equations. This modification has been proposed by using the procedure of Gauss Seidel method and substitution techniques. Initially, each value of unknown or variable has obtained from the corresponding number of equation by arranging it (it means rearrange equation no. 1 for x_1 and rearrange equation no. 2 for x_2 and so on), then these values have been substituted in each equation and rearrange them for variables as Gauss Seidel method. The propose method is used for solving diagonally dominant and positive definite symmetric systems of linear algebraic equations and this method produced very quickly results as compared to Gauss Jacobi's method, Gauss Seidel method and SOR method. The propose method shows the same solution as Gauss Seidel method within the almost half number of iterations of the Gauss Seidel method and same solution as Gauss Jacobi's method within the almost ¼ number of iterations of the Gauss Jacobi's method. This proposed method is best for solving a large number of systems of linear algebraic equations with less number of iterations and such numerical methods have a great importance in solving system of linear equations appearing in the different field of science and engineering. The validity of solution set of proposed method has been tested by comparing the results with Gauss Jacobi's method, Gauss Seidel method and SOR method and it is also verified by using MATLAB software. For this work; a number of problems have been tested and found that the proposed method is a fast convergent method as compared to Gauss Jacobi's method, Gauss Seidel method and Successive over-Relaxation method.

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Funding

This study was conducted independently without financial support from any institutions or sponsors.

Conflicts of interest

The authors confirm the absence of any conflicts of interest associated with this study.

Acknowledgment

The authors express heartfelt appreciation to the institution for their unwavering moral support and encouragement throughout this research endeavor

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