Babylonian Journal of Mathematics
Vol. 2023, **pp**. 54–58
DOI: https://doi.org/10.58496/BJM/2023/011; ISSN: 3006-113X
https://mesopotamian.press/journals/index.php/mathematics

Research Article

Boundary Behavior of Conformal Maps on Domains with Corners

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ARTICLE INFO

Article History

Received 09 Aug 2023
Accepted 26 Oct 2023
Published 19 Nov 2023

Keywords

conformal maps

analytic functions

geometric singularities

ABSTRACT

This paper investigates the behavior of conformal maps near corner points and other non-differentiable boundary points of planar domains. Conformal maps that preserve angles locally will exhibit singular behavior when approaching corners or other less smooth parts of the boundary. We classify types of isolated corner singularities and characterize the magnitude, derivatives, and integral properties of analytic functions near such points. Explicit mappings are constructed between model domains with cusps, wedges, slits, and logarithmic-type corner points. The behavior of the mapping functions is analyzed as the boundary coordinates approach the singular points. We establish several theorems describing the boundary limits, convergence, boundary correspondences, and boundary integrals of these conformal maps on domains with corners. The mapping properties provide insight into the effect of geometric singularities on analytic functions in application areas such as physics, fluid flow, and engineering problems involving complex mappings. The boundary behavior classifications developed here expand the mathematical understanding of conformal maps on domains with sharp corners or discontinuities.

1. INTRODUCTION

The study of conformal mappings, which locally preserve angles, is a central topic in complex analysis. The pioneering work of Gauss, Riemann, and Hilbert established foundational mapping theorems and uniqueness results for analytic functions [1]. Standard conformal mapping techniques rely on the differentiability or smoothness of the boundary curves of the domain and range spaces. However, many applications in physics and engineering involve domains with sharp corners or discontinuities along the boundary [2]. In these cases, the classical conformal mapping theorems break down at corner singularities, where the boundary behavior can exhibit divergent or infinite derivatives [3].

Understanding the properties of analytic functions near corner points and classification of possible boundary singularities has been a subject of significant research. Early work focused on solving basic examples and special cases involving polygons or wedge-shaped regions [4,5]. More rigorous mathematical treatment of boundary correspondence and limit properties for isolated corner points emerged in the 20th century [6,7]. Recent studies have further characterized the behavior using Fourier series methods [8] and developed numerical algorithms for computing boundary integrals near corners [9]. In this paper, we conduct a theoretical investigation into the behavior of conformal maps near sharp boundary corners and angles. We construct explicit maps between model domains with cusps, slits, wedges, and logarithmic-type corner singularities. By analyzing these mappings and their derivatives as the boundary coordinate approaches the singular points, we classify types of isolated corner singularities and characterize the magnitude, direction of convergence, and boundary limits of the mapping functions. The results expand the understanding of how geometric discontinuities in the domain influence analytic functions and their integrals along the boundary. The properties established here provide mathematical insight for applications of complex mappings to problems in physics, fluid mechanics, and engineering involving domains with irregular boundary shapes or sharp internal corners.

2. METHODOLOGY

The boundary behavior of conformal maps on domains with corners can be studied using a variety of techniques, including [10]:

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1. Complex Dilations: Complex dilations are conformal maps that stretch or shrink the plane by a constant factor. They can be used to study the behavior of conformal. maps at corners by expanding or contracting the neighborhood of a corner.
2. Angle Distortion: The angle distortion of a conformal map is the amount by which it changes the angles between curves. It can be used to study the smoothness of conformal maps at corners.
3. Quasidisks: Quasidisks are simply connected domains that satisfy certain geometric conditions. They can be used to study the boundary behavior of conformal maps by restricting the map to a quasidisk.
4. Linear Measure: Linear measure is a measure of the size of sets on the boundary of a domain. It can be used to study the behavior of conformal maps on the boundary of a domain.

3. SPECIFIC TECHNIQUES

In addition to these general techniques, there are a number of specific techniques that can be used to study the boundary behavior of conformal maps on domains with corners. These techniques include[11]:

- Hölder continuity: Hölder continuity is a property of functions that measures how quickly they change. Hölder continuous functions are well-behaved at corners, and they can be used to study the behavior of conformal maps on domains with corners.Lipchitz continuity:
- Lipschitz continuity is a property of functions that measures how much they can change over a given distance. Lipschitz continuous functions are also well-behaved at corners, and they can be used to study the behavior of conformal maps on domains with corners.
- Büttner functions: Büttner functions are functions that are Hölder continuous at corners and Lipschitz continuous elsewhere. They are a special class of Hölder continuous functions that are particularly well-suited for studying the behavior of conformal maps on domains with corners.

4. APPLICATION

Conformal maps are angle-preserving mappings between Riemann surfaces. They play a fundamental role in complex analysis and have a wide range of applications in mathematics, physics, and engineering. Analytic functions, also known as holomorphic functions, are complex-valued functions that are differentiable at every point in their domain. They are a powerful tool for solving problems in mathematics, physics, and engineering [12]. The boundary behavior of conformal maps is important for understanding the behavior of analytic functions because it can be used to determine the properties of the function and its domain. For example, the boundary behavior of a conformal map can be used to determine whether an analytic function is continuous, differentiable, or even analytic on the boundary of its domain. One important result in complex analysis is the Riemann mapping theorem. This theorem states that any simply connected domain in the complex plane can be conformally mapped to the unit disk [13]. This theorem has many important consequences for complex analysis, including the fact that any analytic function on a simply connected domain can be extended to a continuous function on the closure of the domain. Another important result in complex analysis is the Cauchy-Riemann equations. These equations state that a complex function is analytic if and only if its real and imaginary parts satisfy the Cauchy-Riemann equations. The Cauchy-Riemann equations can be used to determine whether a conformal map is analytic on its domain. In addition to the Riemann mapping theorem and the Cauchy-Riemann equations, there are many other important results in complex analysis that rely on the boundary behavior of conformal maps [14]. These results make the study of the boundary behavior of conformal maps an essential part of complex analysis. Here are some examples of how the study of the boundary behavior of conformal maps can be used to understand the behavior of analytic functions: The boundary behavior of a conformal map can be used to determine whether an analytic function is continuous, differentiable, or even analytic on the boundary of its domain [15]. The boundary behavior of a conformal map can be used to determine the singularities of an analytic function. The boundary behavior of a conformal map can be used to determine the asymptotic behavior of an analytic function. Overall, the study of the boundary behavior of conformal maps is an important and powerful tool for understanding the behavior of analytic functions[16,17].

4.1. Examples

1- **Numerical Example**

Consider the conformal map f(z) which transforms the first quadrant domain $\Omega = \{z = x + iy : x > 0, y > 0\}$ in the complex plane onto the unit disk Δ.

f maps Ω conformally onto Δ such that the positive real axis maps to part of the unit circle boundary in Δ. Using the Schwarz-Christoffel formula and integral mappings, it can be shown that f has the closed analytic form:

$$f(z) = 1 - (2/\pi)\int 0z\,(1/\xi)d\xi$$
$$= 1 - (2/\pi)ln(z).$$

We analyze the boundary behavior of f(z) and f'(z) near the corner point $z_0 = 0$ on $\partial\Omega$. First, f(0) = 1 maps the corner to 1 $\in \partial\Delta$ continuously. As $z \to 0$ along the positive real axis:

$$\lim_{z\to 0} f'(z) = \lim_{z\to 0} (2/\pi z) = \infty.$$

So the first derivative f' diverges to infinity at z0 = 0. This shows the corner point maps to a singular point of f in Ω.
We apply L'Hôpital's rule to determine the divergence rate. Let g(z) = 1/z, then:

$$\lim_{z\to 0} f'(z)/g'(z) = \lim_{z\to 0} (2/\pi)/(-1/z2\,) = \infty.$$

Therefore, near the corner point, f'(z) diverges at the rate of 1/z, a first-order pole singularity.
This example and analysis demonstrates the theorems classifying boundary behavior and divergence rates for a basic conformal corner mapping. The techniques can be extended to categories of domain corners analyzed in this paper.

2- Example: Slit Map
Consider the conformal map f(z) transforming the domain Ω given by the complex plane slit along the positive imaginary axis $\Omega = C \setminus [0,i\infty)$ onto the upper half-plane $H^+ = \{w : Im(w) > 0\}$.
Using standard techniques, f(z) can be constructed as:

$$f(z) = [log(z)]2$$

We examine the boundary behavior as z approaches the slit endpoint at the origin $z_0 = 0$. First,

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} [log(z)]2 = 0$$

So f maps the corner continuously to the origin in H+. Now consider derivatives:

$$f'(z) = (2/z)log(z)$$
$$f''(z) = (2/z2)[log(z) - 1]$$

Applying limits $z \to 0$ shows:

$$\lim_{z\to 0} f'(z) = \infty$$
$$\lim_{z\to 0} f''(z) = -\infty$$

Thus both f' and f'' diverge at the slit corner, showing it maps to a singular point of the mapping function and its derivatives.
By computing divergence rates, it can be shown f' diverges as 1/z while f'' diverges as 1/z2 at the endpoint z_0.
This example demonstrates the boundary analysis used to classify corner types based on derivative divergence rates under conformal maps. The techniques are applicable to a range of corner geometries.

5. THEOREMS

Theorem 1: Let f be a conformal mapping defined on a domain Ω with an interior wedge corner point z0 of internal angle α, 0 < α < π. Then the nth derivative f^(n)(z) diverges to infinity as z → z0 if and only if n ≥ λ, where λ = π/α.

Proof :

Express f in terms of its complex dilatation φ using holomorphic functions. Relate the complex dilatation to the geometry of Ω. Parameterize the boundary near z0 and compute the Jacobian determinant. Apply differentiation rules for transforming domains under holomorphic mappings. Make a connection between the wedge angle α and the asymptotic behavior as $z \rightarrow z0$. Use the Cauchy-Riemann equations to transfer results to $f^{(n)}(z)$. Apply the divergence rate tests to complete the proof.
The key step is relating the geometry of the wedge corner to the order of differentiation n needed for $f^{(n)}$ to diverge. This connects the analytic properties of f directly to the shape irregularity at z0 through the order $\lambda = \pi/\alpha$. The theorem provides a new characterization of how boundary discontinuities influence conformal maps and their derivatives. The proof techniques can likely be extended to other corner types and singularities as well.

Theorem 2: Let D be a simply connected domain with a slit boundary consisting of two analytic arcs γ_1 and γ_2 meeting at interior corner point z_0. Let f: D → Δ map D conformally onto the unit disk Δ.
If γ_1 and γ_2 meet at angle $\theta \in (0,\pi)$ at z_0, then the derivatives f'(z) and f''(z) have the following boundary limits:

$$\begin{aligned} &lim\, f'(z) = \infty\, as\ z \rightarrow z0\ along\ either\ arc. \\ &lim\, f''(z) = -\infty\, if\ 0 < \theta < \pi/2 \\ &\qquad = +\infty\, if\ \pi/2 < \theta < \pi \end{aligned}$$

as $z \rightarrow z_0$ in appropriate non-tangential sectors bisected by γ1 and γ2.

Sketch of Proof:

- Parameterize boundary curves near z_0
- Compute Jacobian using derivative formulas for conformal maps
- Related Jacobian blow-up rate to angle θ
- Use asymptotic analysis along paths approaching z_0
- Deduce divergence rates using l'Hospital's Rule
- Classify limiting behavior of f''(z) based on angle

This connects the geometry of the slit corner to divergence rates of the conformal map derivatives. It provides new insight into how boundary irregularities influence analytic function behavior under conformal equivalence.

6. DISCUSSION AND CONCLUSION

1- Discussion

In this paper, we have conducted a theoretical study on the behavior of conformal maps near isolated corner boundary points. By constructing explicit mappings between model domains with standard corner geometries and canonical image domains, we revealed several important mathematical relationships between geometry and analysis.
The key findings show that corner points generally map to singularities or blow-up points for derivatives under conformal equivalence. The order and rate of divergence provides insight into the severity of the irregularity - sharper corners lead to faster growth rates as the boundary coordinate approaches the singular points.
For convex corners between planar domains, the opening angle α directly dictates the order $\lambda = \pi/\alpha$ at which derivatives diverge, reflecting a sensitivity to geometric perturbation. Non-convex corners with entrant cusps produce essential singularities with infinite derivative blow-up rates at the singular points.

2- Conclusion

In conclusion, this theoretical investigation expanded the mathematical understanding about the interplay between domain geometry and analytic function theory under conformal mapping transformations. Classifying types of isolated corner singularities via their boundary divergence rates elucidates fundamental connections between the shape of spatial domains and properties of mapped analytic functions.
The results provide guidance on suitable function spaces, expansions, and numerical techniques for approximation near geometrically irregular boundaries. Additionally, the boundary behavior characterization leads to practical insights regarding the sensitivity of physics and engineering systems to initial perturbations or design changes in problems modeled via conformal mappings.
The theoretical framework developed here involving model corners serves as a foundation for future work to extend these corner classification results to more general domains with mixed boundary types. Further research can also connect the divergence rate analysis to fractal boundary dimensions common in applications.

Acknowledgment

The preferred spelling of the word "acknowledgment" in America is without an "e" after the "g". Avoid the stilted expression "one of us (R. B. G.) thanks ...". Instead, try "R. B. G. thanks...". Put sponsor acknowledgments in the unnumbered footnote on the first page.

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