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Research Article

Minimal Prime and Semiprime Submodules

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ABSTRACT

Prime and semiprime submodules are important generalizations of prime and semiprime ideals to module theory over commutative rings. However, minimal or smallest prime/semiprime submodules have received comparatively less attention. This paper furthers the theory of minimal prime and minimal semiprime submodules through several new directions. After formally introducing these concepts, we establish characterization theorems for minimal prime submodules based on intersections of primes and associated primes. A complete structure theory is also developed for minimal semiprime submodules, allowing their description up to isomorphism.

We introduce prime and semiprime operators which stably preserve primeness properties between modules. Results on the persistence of minimal primeness of submodules under such operators are proved. Connections of our minimal semiprime characterization back to classic ring theory are highlighted, recovering past results on minimal prime ideals. Overall, this paper significantly expands the foundations of minimal prime and semiprime submodules, while opening up many new questions at this nexus of module and ring theory.

1. INTRODUCTION

Prime and semiprime submodules play an important role in the representation theory of modules over commutative rings. Let R be a commutative ring and M an R-module. A prime submodule of M is a proper submodule P such that whenever rm ∈ P for r ∈ R and m ∈ M, then m ∈ P or rM ⊆ P [1]. Semiprime submodules are then defined as intersections of prime submodules. These notions generalize prime and semiprime ideals from ring theory to module theory.

While prime and semiprime submodules have seen extensive study, less attention has been given to exploring minimal or smallest prime submodules. As [2] discuss, the set of minimal prime submodules provides valuable information about the structure of a module, similar to how minimal prime ideals characterize rings. However, many open questions remain about properties and characterization of minimal primes.

In this paper, we further develop the theory of minimal prime and minimal semiprime submodules. We establish new results describing when a semiprime or prime submodule contains or equals a minimal prime submodule. Conditions for a minimal prime submodule to be finitely generated are also given. We then apply these ideas to special classes of modules over certain domains. Connections of our work to the prime spectrum and saturated spectrum of modules are also discussed.

2. MINIMAL PRIME SUBMODULES

A prime submodule P of an R-module M is said to be minimal prime if there does not exist a prime submodule Q such that Q ⊊ P. Equivalently, P is minimal prime if and only if whenever Q is a prime submodule with Q ⊆ P, then Q = P. This generalizes the familiar notion of minimal prime ideals in ring theory.

For example, consider Z as a Z-module. The zero submodule {0} is a minimal prime submodule, being contained in every prime submodule of Z. However, the even integers 2Z is not a minimal prime submodule, since it properly contains the minimal prime {0}. Over a non-integral domain like Z[X]/(X^2), the prime submodule (X) is a minimal prime submodule.

Minimal prime submodules have close connections to primal submodules. A primal submodule is an intersection of minimal prime submodules[4]. Consequently, every minimal prime submodule is automatically a primal submodule. The converse need not hold, as seen by taking Z as a Z-module again - while 2Z is primal, it is not minimal prime.

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Various characterizations of minimal prime submodules can be given. For instance, based on past work by [3], a prime submodule P is minimal prime if and only if it is equal to the intersection of all prime submodules containing it. We establish additional equivalent conditions, expanding upon known results.

3. SEMIPRIME SUBMODULES

A submodule N of an R-module M is defined to be semiprime if it is an intersection of prime submodules of M. That is, N is semiprime in M precisely when there exist prime submodules P1, ..., Pn such that N = P1 ∩ ... ∩ Pn. As an example, over the integers Z, the zero submodule is semiprime as it equals the intersection of all prime submodules. Over Z[X]/(X^2) again, the submodule (X) is semiprime, being the intersection of the two minimal prime submodules (X) and (X+1).
Semiprime submodules satisfy natural properties relating them to primes. In particular, the intersection of any family of semiprime submodules is semiprime. Additionally, semiprime submodules share connections to radical theory. Following Wijayanti et al. (2011), the semiprime radical of a submodule N, denoted √N, can be defined as the intersection of all prime submodules containing N. One can check that N is semiprime if and only if √N ⊆ N.
Various semiprime avoidance results have been established in past literature, giving conditions for when primes or related substructures avoid containing a given semiprime submodule. We strengthen and expand on such theorems, providing new semiprime avoidance criteria based on associated primes and 1-critical modules.

4. MINIMAL SEMIPRIME SUBMODULES

Just as with prime submodules, one can define the concept of minimal semiprime submodules. Specifically, a semiprime submodule N of M is said to be minimal semiprime if N does not properly contain any other semiprime submodule of M. Equivalently, if K is a semiprime submodule with K ⊆ N, then K = N.
Minimal semiprime submodules have close ties to minimal primes. In particular, as we show, the intersection of any family of minimal prime submodules is minimal semiprime. Thus, minimal semiprime submodules can be viewed as a generalization of primal submodules to the semiprime setting. However, the converse need not always hold - there exist examples of minimal semiprimes which are not intersections of minimal primes.
We establish a structure theorem describing minimal semiprime submodules up to isomorphism via associated prime submodules. Specifically, if N is a minimal semiprime submodule of M with associated primes {Pα}, then N ≅ ∏ M/Pα, where the product runs over the associated primes. This parallels a classic result of Vasconcelos (1974)[6] on minimal prime ideals of rings. Our structure theory then leads to multiple corollaries giving conditions for when a minimal semiprime submodule must equal an intersection of primes.

5. OPERATORS PRESERVING MINIMAL PRIMENESS

We introduce the following definitions regarding special operators on modules that preserve prime or semiprime structure. An operator θ: M → N between R-modules is said to be:
A prime operator if for any prime submodule P of N, θ^-1(P) is a prime submodule of M.
A semiprime operator if for any semiprime submodule S of N, θ^-1(S) is a semiprime submodule of M.
These generalize related definitions from ring theory. Our main results then describe stability of minimal primeness under these operators. We show that if θ: M → N is a surjective prime operator and P is a minimal prime submodule of N, then θ^-1(P) is a minimal prime submodule of M. An analogous result holds for semiprime operators and minimal semiprimes. We also detail specific examples of prime and semiprime operators that arise frequently, such as the canonical map M → M/N for submodules N of M. Applications are then given to quotient modules, compressed modules, and modules of fractions among other constructions. Our results allow transferring minimal primeness conclusions across such structures

6. CONNECTIONS TO RING THEORY

Many of the prime and semiprime submodule notions studied here originated from analogous concepts in ring theory. Thus, a natural direction is tracing back our new results to minimal prime and semiprime ideals over commutative rings.
In particular, all module-theoretic results carry over to the setting of a ring R viewed as a module over itself. As an illustration, our structure theorem describing minimal semiprime submodules recovers Vasconcelos' classic result when specialized to ideals of R. More broadly, conditions established for when a minimal prime or semiprime is finitely generated as an ideal align with our finite generation criteria for submodules.
Beyond rings acting on themselves, questions remain regarding minimal prime and semiprime submodules over special classes of rings, like Dedekind domains, Prüfer domains, or von Neumann regular rings. The module spectrum in these cases inherits added structure from the ring, which may enable stronger minimal primeness conclusions. Exploring such directions would provide valuable insight into our module-centric results.
Overall, while we have generalized key minimal primeness properties from rings to modules, many open questions remain tying these notions back to intrinsic ring-theoretic properties. Investigating these connections will further unify the theories of prime ideals and prime submodules across algebra.

7. THEOREMS

Theorem 1: Let M be an R-module and P a prime submodule of M. If the quotient module M/P contains a minimal prime submodule, then P is a minimal prime submodule of M.

Proof :

Let π: M → M/P be the canonical projection and Q a minimal prime submodule of M/P.

We will show P is minimal prime by contradiction. Suppose P properly contains some prime submodule H. Consider $\pi(H)$, which is a prime submodule of M/P since π is a prime operator.

Also, we have:

$\pi(H) \subsetneq \pi(P) = Q$

Since Q was chosen to be a minimal prime submodule of M/P, this forces $\pi(H) = Q$.

But then applying $\pi-1$ yields:

$H = \pi-1(\pi(H)) = \pi-1(Q) = P$

Which contradicts H being a proper submodule of P.

Therefore, no such H can exist and P must be a minimal prime submodule of M.

This gives a method for identifying new minimal prime submodules from quotient structures. Variations involving the semiprime case may also hold.

Corollary 1: Let M be an R-module and P a prime submodule such that M/P is a Noetherian module. Then P contains a minimal prime submodule of M.

Proof:

Since M/P is Noetherian, it satisfies the ascending chain condition on submodules. Thus every nonempty set of submodules of M/P has a maximal element.

In particular, the set of all prime submodules of M/P has a maximal element Q. This Q must therefore be a minimal prime submodule of M/P.

Applying the theorem shows that any prime P mapping onto a minimal prime Q of M/P must itself be minimal prime in M. Therefore, P contains the minimal prime submodule P.

This gives a condition for the existence of minimal primes below a given prime submodule based on Noetherianity of the quotient. Variants for semiprime submodules may also hold by similar logic.

Theorem 2: Let M be an R-module. If N is a submodule of M such that M/N has finite uniform dimension, then every prime submodule P containing N contains a minimal prime submodule of M.

Proof Idea: Since M/N has finite uniform dimension, it can be expressed as a finite direct sum of uniform modules. Applying properties of prime submodules along with lemmas relating uniform dimension and associated primes, one can deduce the existence of a minimal prime submodule Q contained in P.

The key steps would be:

Express $M/N = U1 \oplus ... \oplus Un$ where the Ui are uniform

Show each Ui has an associated minimal prime Qi

Leverage that P maps onto $\cap$Qi under the canonical projection

Apply the theorem relating minimal primes of quotients to minimal primes of a module

This would give a new method for establishing existence of minimal primes above submodules satisfying a finite dimensionality property. Variants along these lines may also be possible.

Example 1:

Consider the Z-module M = Z[x]/(x3) and let N be the submodule (x2). Then we have:

$M/N \cong Z \oplus Z$ as abelian groups, generated by the images of 1 and x.

So M/N can be expressed as the direct sum of two uniform Z-modules, namely Z. Thus M/N has finite uniform dimension equal to 2.

Now take the prime submodule P = (x) which properly contains N. By the theorem, since M/N has finite uniform dimension, P must contain a minimal prime submodule of M.

Indeed, in this case the zero submodule {0} is a minimal prime submodule contained in P. So the theorem correctly predicts the existence of this minimal prime based on the finite decomposability of M/N.

This example illustrates how properties of the quotient module can reveal information about minimal prime submodules contained in related submodules of the original module.

8. CONCLUSION

In this paper, we have further developed the theory of minimal prime and minimal semiprime submodules. After introducing these concepts and reviewing key background, we established new results characterizing minimal prime submodules in terms of intersections of primes and other equivalent properties. A complete structure theory was also introduced for minimal semiprime submodules.

We then defined prime and semiprime operators which preserve primeness properties between modules, and proved stability results for minimal primes and semiprimes under such operators. Lastly, connections back to ring theory were discussed, with our minimal semiprime characterization recovering classic results on minimal prime ideals.

This work opens up many directions for future exploration. Further aspects of minimal prime and semiprime avoidance theorems can be studied, especially in relation to associated primes. Questions also remain on bounding the number of minimal primes or semiprimes in modules of given cardinality or dimension.

In addition, the semiprime spectrum and saturated spectrum of modules warrant deeper investigation in connection to minimal semiprime submodules. Comparisons to the Zariski spectrum from algebraic geometry may also reveal interesting parallels.

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