Babylonian Journal of Mathematics
Vol. **2024**, **pp**. 48–52
DOI:https://doi.org/10.58496/BJM/2024/006; ISSN:3006-113X
https://mesopotamian.press/journals/index.php/mathematics

Research Article

Restricted visible submodules and fully restricted visible module

Mahmood S. Fiadh[1,*], Buthyna N. Shihab [2], Ahmed Issa[3]

[1]*Dep. of computer Sci., College of Edu., AlIraqia University, Baghdad, Iraq.*
[2]*Dep. of Math., College of Edu. for Pure Sciences Ibn AL-Haitham, Univ. of Baghdad, Baghdad, Iraq.*
[3]*Dep. of Math., faculty of science, Karabük univ., Turkey.*

ARTICLEINFO

Article History
Received 01 Jan 2024
Revised 19 Mar 2023
Accepted 02 Apr 2024
Published 24 Apr 2024

Keywords
Fully visible module
fully restricted visible module
Math
Multiplication module
Duo module

ABSTRACT

In this article, the concept of Restricted visible (for short; Res-visible) submodule and fully restricted visible (for short fully Res-visible) module have been introduced which are considered the generalizations of the concepts of visible submodules and fully visible module respectively where every visible submodule (fully visible module) is Res-visible submodule (fully Res-visible) module, but the converse is not be true. Examples have been presented illustrating those relationships.

1. INTRODUCTION

In this study, T is a commutative ring with identity and X is unitary module. Anderson and Fuller [1] called the submodule N a pure submodule of X if $IN = N \cap IX$ for every ideal I of T. Ribenboim [7] defined N to be pure in M if $rX \cap N = rN$ for each $r\epsilon\, T$. Mijbass in 1992 presented the concept of cancellation module "A T-module X is defined to be a cancellation module if $IX = JX$ for ideals I and J of T implies I=J ", [6].

In [2] Mahmood and buthyna presented a new type of submodule of a module X over a ring T under the name visible submodule. Where a proper submodule W of a T -module X is named visible whenever $W = IW$ for every a nonzero ideal I of T , also in ([5],[4]) they present the concepts of 2-visible module and fully visible module where a submodule K of a module X over a ring T is said to be 2-visible whenever $K = I^2K$ for every nonzero ideal I of T, a module X over T is called a fully visible module , if each a proper submodule of T is visible, our aim goal is to introduce the concept of Restricted visible (for short; Res-visible) submodule and fully restricted visible (for short fully Res-visible) module .

2. RESTRICTED VISIBLE SUBMODULES

The concept of restricted visible (for short, Res-visible) submodule of a T-module X will be presented here in this item which is a generalization of the concept of visible submodule of an T-module X. It is clear that each visible submodule is restricted visible submodule, but the reverse does not occur, and we have given an example that illustrating it. Many results and properties that the researcher can observe here have been demonstrated.

Let us begin with our basic definition:

2.1 Definition

A nonzero proper submodule W of a module X over T is named restricted visible, if for every nonzero ideal I of T such that $IN \neq 0$, implies $N = IN$.

*Corresponding author. Email: salim_fa@yahoo.com

2.2 Remarks and examples

Let $T = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a, b \in Z_2\}$ be the ring 2×2 matrices over the field Z_2. Then $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ and $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\}$ are two proper nonzero left ideals of T with the trivial ideal $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$. Let $X = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, a, b \in Z_2\}$ be a module over T and Let $N = \{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in Z_2\}$ be a submodule of X. Thus N is Res-visible submodule of X. Since , if we take the ideal $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$.

We get $N = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} N = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If we take the ideal $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\}$, we obtain $N = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\} N \neq 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Also $N = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} N \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. From all above, we have satisfied that N is Res-visible sumodule.

The submodule $(\bar{2})$ of a Z_8-module Z_4 is Res-visible. Since $(\bar{1})$ is the only ideal of Z_8 such that $(\bar{1})(\bar{2}) \neq 0$ and implies $(\bar{1})(\bar{2}) = (\bar{2})$.

More generally, every submodule of a Z_{2^n} –module Z_4 , where n is any positive integer and $n \geq 2$ is Res-visible submodule.

Consider the module Z_6 over Z. Since the two submodules $(\bar{2})$ and $(\bar{3})$ are not visible to show that, take $(\bar{2})$. Note $(5)(\bar{2}) \neq 0$, but $(5)(\bar{2}) = (\bar{4}) \neq (\bar{2})$. Therefore, $(\bar{2})$ is not Res-visible submodule. In the same way, when we take the submodule $(\bar{3})$.

Every submodule of a Z-module Z is not Res-visible, also all submodules in the module $Z \oplus Z$ over Z are not Res-visible. To prove this, let $W = nZ \oplus mZ$ be a submodule of $Z \oplus Z$, n, m are a nonzero positive integer and $n, m > 1$. Let $3Z$ be an ideal of Z, since $(3Z)W = 3Z(nZ \oplus mZ) = (3n)Z \oplus (3m)Z \neq (0,0)$, but $(3Z)W \neq$. That is not Res-visible submodule.

Any submodule of a Z_P-module Z_P and all submodules of Z-module $Z \oplus Z$ are not Res-visible.

Every visible submodule is Res-visible. The opposite incorrect and the following example illustrates this: the submodule $(\bar{2})$ of Z_4 as a Z-module is Res-visible by number (2) , but it is not visible submodule by (Remarks and Examples ((2.2),(1)).

Theorem 1: Let W and D be two Res-visible submodules of a T-modules X. Then $W + D$ is also Res-visible submodule.

Proof : we start with W and D are two Res-visible submodules of X and let J be a nonzero ideal of $T$and $J(W + D) \neq 0$. If $(W + D) = 0$, then JW and JD are equal to zero and this contradiction with the fact that W and D are Res-visible submodules. To get back now,

$J(W + D) = JW + JD = W + D$. This prove to be.

More general, if $\{W_k\}_{k=1}^{n}$ is a finite collection of a Res-visible submodules of an T-module X, then the sum of all these submodules are Res-visible.

Theorem 2: Suppose that L be a Res-visible submodule of X. Then for every nonzero proper submodule K of X is Res-visible whenever $L \cong K$.

Proof : assume that $\hat{\rho}: L \longrightarrow K$ be an isomorphism. Then $\hat{\rho}\,(L) = K$ (since $\hat{\rho}$ is an epimorphism), but L is Res-visible submodule of X, then $L = IL$, for every nonzero ideal I of T such that $IL \neq 0$.

$K = \hat{\rho}\,(L) = \hat{\rho}\,(IL) = I\hat{\rho}\,(L) = IK$. Since $IK \neq 0$, if $IK = 0$, then $\hat{\rho}\,(IL) = 0 = f(0)$ and hence $IL = 0$. (since $\hat{\rho}$ is monomorphism), but this is a contradiction with assumption. Therefore $IK \neq 0$. Thus K is Res-visible.

Theorem 3: Let X_1 and X_2 be T-modules and $\hat{\gamma}: X_1 \longrightarrow X_2$ be an T-homomorphisim. Then $\hat{\gamma}^{-1}(L)$ is Res-visible submoule of X_1 when L is Res-visible submodule of X_2.

Proof : let L be a Res-visible submoule of X_2. Then $L = IL$ for every nonzero ideal I of T such that $IL \neq 0$.

Now $\hat{\gamma}^{-1}(L) = \hat{\gamma}^{-1}(IL) = I\hat{\gamma}^{-1}(L)$, since $I\hat{\gamma}^{-1}(L) \neq 0$.

If $I\hat{\gamma}^{-1}(L) = 0$, then $\hat{\gamma}^{-1}(IL) = 0$ and hence $\hat{\gamma}\hat{\gamma}^{-1}(IL) = \hat{\gamma}(0) = 0$ which implies $IL = 0$. This gives us a contradiction. So it must be $I\hat{\gamma}^{-1}(L) \neq 0$ and from this we get this we get $\hat{\gamma}^{-1}(L)$ is Res-visible submoule of X_1.

The next proposition gives an advantage to the Res-visible submoule.

2.3 Proposition

Let W be a nonzero proper and faithful submodule of a module X over T. Then we will get the next bonuses.

1) W is Res-visible submodule.

2) $W = IW$ for every nonzero $\mathrm{f.g}$ ideal I of T such that $IW \neq 0$.

Proof:

(1) ⟹ (2) directly realized.

$(2) \Longrightarrow (1)$ Let W is Res-visible submodule of X. Consequently $\forall I \neq 0$, I is an ideal of T , we have $W = IW$, we can take I is finitely generated ideal.

we have to prove that $IW \neq 0$ for every nonzero $\mathrm{f.g}$ ideal I of T. If $IW = 0$, then $I \subseteq ann(W) = 0$ (since W is faithful) which implies $I = 0$ this is a contradiction. Therefore $IW \neq 0$.

The following proposition confirms the genetic property of Res-visible submoule.

2.4 Proposition

Every submodule of Res-visible submodule is also Res-visible.

Proof:

Similarly proof of (Proposition (2.7),[2])

2.5 Corollary

Let W_1 and W_2 be two Res-visible submodules of a T-module X, then $W_1 \cap W_2$ is Res-visible submodule.

As a direct generalization of the result above we gave the following

2.6 Corollary

Let $\{W_i\}_{i=1}^{n}$ be a family of Res-visible submodules of aT-module X. Then $\bigcap_{i=1}^{n} W_i$ is Res-visible submodule.

2.7 Remark

Let K, W be two submodules of a T-modules X such that $K \subset W$ and K is Res-visible submodule, it is not necessary that W is Res-visible submodule and the following illustrates this. Since $(\bar{2})$, $(\bar{4})$ are two submodules of Z_{12} and $(\bar{4}) \subseteq (\bar{2})$. Note $(\bar{4})$ is Res-visible submodule, but $(\bar{2})$ is not Res-visible submodule look, $(\bar{2}).(\bar{3}) = (\bar{6}) \neq (\bar{0})$ where $(\bar{3})$ is an ideal of Z_{36} and hence $(\bar{2}).(\bar{3}) \neq (\bar{2})$ and this explained above.

2.8 Proposition

Let X be a fully cancellation module. Over a ring T in which all nonzero ideals are idempotent and K is a nonzero proper submodule of X. If D is a Res-visible submodule and contain in K, then K is Res-visible submodule.

Proof:

Let K be a nonzero proper submodule of X and D be a submodule such that $D \subseteq K$. Then for every nonzero ideal I of T, we have $ID \subseteq IK$ and hence $IK = ID + IK$, but $D \subseteq IK$ (since D is Res-visible submodule). Therefore $IK = D + IK$ (since $ID = D$, Res-visible submodule). Then $IK = IK$. We have, $I^2K = I^2K$, implies $IK = I^2K$ (since I is an idempotent ideal), but X is fully cancellation module, then $K = IK$ since $IK \neq 0$, if $IK \neq 0$, then $ID = 0$ (note, $ID \subseteq IK$), this is a contradiction because D is Res-visible submodule.

3. FULLY RESTRICTED VISIBLE MODULE

Now, we will introduce the definition of fully restricted visible (for short, fully visible) modules which is a generalization of the concept of fully visible module. So let's start by definition.

3.1 Definitions

A T-module X is said to be fully Res-visible if for any nonzero proper submodule of X is Res-visible. A ring T is called fully Res-visible if it is fully Res-visible T-module.

The next gives us examples and observation about that concept.

3.2 Examples and Remarks

Z_4 as a Z_4-module is fully Res-visible. 1

This property can be written number more generally as follows: Z_4-module is fully Res-visible Z_{2^n}-module where $n > 2$ by Remarks and Examples ((4.1.2), (3)).

Z_6 as a Z_6-module is not fully Res-visible since, let $(\bar{2})$ be a submodule of Z_6 and $(\bar{1}), (\bar{2}), (\bar{3})$ are ideals of a ring Z_6then $(\bar{1}).(\bar{2}) \neq (\bar{0})$ and $(\bar{1}).(\bar{2}) = (\bar{2})$ also, $(\bar{2})(\bar{2}) \neq (\bar{0})$.

But $(\bar{2}).(\bar{2}) = (4) \neq (\bar{2})$ that is, $(\bar{2})$ is not Res-visible submodule of Z_6. This prove the result.

Z_P as a Z_P, P is prime number is not fully Res-visible module.

Z as a Z-module is not fully Res-visible, since every submodule of Z is not Res-visible also, the module $Z\oplus Z$ as a Z-module is not fully Res-visible by examples and remarks (2.1) number (2).

Every fully visible is fully Res-visible, since the opposite is not true in general for example: let's take the same example as in ((4.1.2), (1)) it is fully Res-visible module, since the submodule $W = \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, a \in Z_2\right\}$ of a module $X = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, a, b \in Z_2\}$ of a ring $\mathrm{T} = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a, b \in Z_2\}$. Let $I = \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ be a left ideal of X. Then $W \neq IW = 0$. W is not visible module of X and hence X is not fully visible module.

- Every submodule of fully Res-visible module over T is fully Res-visible.

Let $\psi: X_1 \longrightarrow X_2$ be T-homorphism and X_1 , X_2 be two fully Res-visible T-modules. Then :

i.If W is a nonzero proper submodule of X_1, then $f(W)$ is Res-visible submodule of X_2.

ii.If $W`$ is a nonzero proper submodule of $X_2$ then $f^{-1}(W`)$ is Res-visible submodule of X_1.

The following theorem is a generalization of (Theorem (2.1.4), [3])where it gives advantages to the module to be fully Res-visible.

3.3 Theorem

Let X be a T-module. Then the following are equipollent.

- X is fully Res-visible module.
- Every nonzero proper cyclic submodule of X is Res-visible.

3.4 Proposition

Let T be a PIR. and W be a nonzero proper submodule of a T-module X. If for all $x \in N$ and $\forall 0 \neq r \in T$ such that $x = rtx$ for some $t \in T$, then X is fully Res-visible module.

Proof:

Similar the proof of (Proposition (2.1.5),[3])

3.5 Proposition

If X is fully Res-visible module over T, W is a nonzero proper submodule of X. Then for all $x \in W$ and for all $s \neq 0 \in T$, we have $x = stx$ for some $t \in T$.

Proof:

Suppose that $x \in W$ and $0 \neq s \in T$. Hence (s) is an ideal of T. Now, we have X is fully Res-visible module, then $W = IW$ for every nonzero ideal I of T. Put $I = (s)$. Hence $x \in W = (s)W$. Since $(s)W \neq 0$, if $(s)W = 0$, then $W = 0$, this is contradiction with the W hypothesis.

3.6 Corollary

Let X be a module over PIR and W be a nonzero proper submodule of X. Then X is fully Res-visible module if and only if for all $x \in W$ and for all $0 \neq r \in T; x = rxt$ for some $t \in T$.

Funding

The author's paper clearly indicates that the research was conducted without any funding from external sources.

Conflicts of interest

The author's disclosure statement confirms the absence of any conflicts of interest.

Acknowledgment

The author extends appreciation to the institution for their unwavering support and encouragement during the course of this research.

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, 1974.
[2] M. S. Fiadh and B. N. Shihab, "Visible submodules of a module X over a ring R is introduced," Journal for Engineering and Applied Sciences, vol. 13, no. 24, pp. 10349-10355, 2018.
[3] M. S. Fiadh, "Visible (W-Visible) Submodules and Fully Visible (W-Fully visible) Modules With Some of Their Generalizations," Ph.D. dissertation, Univ. of Baghdad, Baghdad, Iraq, 2019.
[4] M. S. Fiadh and B. N. Shihab, "Fully visible modules with the most important characteristics," Journal of Discrete Mathematical Sciences and Cryptography, vol. 25, no. 5, 2022.
[5] M. S. Fiadh and W. H. Hanoon, "2-Visible Submodules and Fully 2-Visible Modules," Iraqi Journal for Computer Science and Mathematics, vol. 1, no. 2, pp. 24-28, 2020.
[6] A. S. Mijbass, "On cancellation modules," M.Sc. thesis, Univ. of Baghdad, Baghdad, Iraq, 1992.
[7] P. Ribenboim, Algebraic Numbers, Wiley, 1972.