



Research Article

# Nearly Endo Quasi Prime Sub-modules

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## ABSTRACT

In this paper, the notion of nearly Endo quasi prime sub-modules is introduced together with several fundamental features and theorems. It provides several properties and descriptions of these ideas, and additional diagrams and examples are used to show how the study's findings translate into nearly Endo quasi prime and nearly Endo prime sub-modules

## 1. INTRODUCTION

$G$  is would be a commutative ring and the identity in this work. Lu[1] conducted research on the idea of a prim sub-modules of modules in 1983 as a generalization of the idea of the prime ideal, while Ahmad Y. D. and Fatemeh S. introduced the idea of the 2-Absorbing sub-module in [2] as a generalization of the idea of the prime sub-module. Abd Ali and Hanoon [6] established the concepts of NEndo T-ABSO sub-modules and NEndo prime sub-module. In this article, the notion of Endo quasi prime sub-module 1.is extended to NEndo quasi prime sub-module. Section 2 examines the NEndo quasi prime sub-modules and all of its significant features, conclusions, and results.

## 2. PRELIMINARIES

The several basic ideas are covered in this section, along with any requirements they could have on the area that follows.

**Definition 1 :** A sub-module  $S \leq V$  is called to *min.* (accordingly *max.*) sub-module of  $V$  if  $S \neq 0, \forall K \leq V, K \subsetneq S \Rightarrow K=(0)$  [accordingly  $K \not\subseteq V, \forall S \leq V, S \subset K \Rightarrow K=V$ ] [7]

**Definition 2 :**  $G$  module  $V$  is called to *a cyclic* if  $h \in V$  suchthat  $V = \langle h \rangle = \{rh : r \in G\}$ . [7]

**Definition 3 :** If a module  $V$  has finite generating set, It's said that finitely generated., say  $S$ , that is  $V = \langle S \rangle$ . [9]

**Definition 4 :** A sub-module  $S$  of a  $L$  – module  $V$  is called to as a direct summand of  $V$ , for short  $S \leq^{\oplus} V$  if, there exists a sub-module  $K$  of  $V$  such that  $S + K = V$  and  $S \cap K = 0$ . [8]

**Definition 5:** Let  $V$  as  $L$ -module and  $S \subset V$ .  $A$  is called to as a prime sub – module if  $r \in L, h \in V, \text{with } hs \in S$  implies that  $h \in S$  or  $r \in (S :_L V)$ . [1]

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**Definition 6:** Let  $V$  as  $L$  – module and  $S \subset V$ .  $P$  is called to as Endo Prime sub – module if  $f \in \text{End}(V)$ ,  $f(h) \in S, h \in V$  implies that  $h \in S$  or  $f(V) \subseteq S$ . [ 3 ]

**Definition 7:** Let  $V$  as –module ,  $S \subset V$ .  $S$  is called as TABSO sub – module if whenever  $x, y \in L, a \in V$ , with  $xya \in S$  implies that  $ax \in S$  or  $ya \in S$  or  $ab \in (S;_L V)$ . [2]

**Definition 8 :** Let  $V$  as  $L$  – module and  $S \subset V$ .  $S$  is called to as Endo TABSO sub – module if  $\forall J, H \in \text{End}(V)$ ,  $m \in V$  with  $(J \circ H)(m) \in S$  implies that  $J(m) \in S$  or  $H(m) \in S$  or  $(J \circ H)(V) \subseteq S$ . [5]

**Definition 9:** Let  $V$  be a  $L$  – module. The Jacobson radical of  $M$  is indicated by  $J(V)$ , and defined as all max. sub-modules of  $V$  intersecting, and indicated by sum of all small sub-module of  $V$ . If  $V$  has no max. sub-module, then we set  $J(V) = V$ . [7]

**Theorem 1 :** If  $f: V \rightarrow V'$  is a  $L$  – homomorphism, then  $f(J(V)) \subseteq J(V')$ , If  $f: V \rightarrow V'$  is a  $L$  – epimorphism and  $\ker f \ll V$ , then  $f(J(V)) = J(V')$ , and  $J(V)L \subseteq J(V)$ , wher  $L$  is a ring, if  $V$  is projective module then  $J(V)L = J(V)$ . [7]

**Definition 10 :** Let  $V$  as  $L$  – module and  $S \subset V$ ,  $S$  is called to as NEndo prime sub-module if  $\forall f \in \text{End}V, x \in V$  such that  $f(x) \in S$  implies that  $x \in S + J(V)$  or  $f(V) \subseteq S + J(V)$ . [6]

**Definition 11:** Let  $V$  as  $L$  – module and  $S \subset V$ ,  $S$  is called to as NEndo TABSO sub-module if  $\forall J, H \in \text{End}V, x \in V$  such that  $(J \circ H)(x) \in S$  implies that  $J(x) \in S + J(V)$  or  $H(x) \in S + J(V)$  or  $(J \circ H)(V) \subseteq S + J(V)$ . [6]

**Definition 12 :** A  $L$  – module  $V$  is called to as a scalar module if for each  $J \in \text{End}(V)$ ,  $\exists n \in L$  such that  $J(x) = nx$ , for  $x \in V$ . [9]

**Corollary 1** Every finitely generated multiplication  $L$ -module  $V$  is scalar module[9].

**Definition 13:** A  $L$ -modul  $P$  is called to as  $W$  – Projective module if each schematic diagram [ 7 ] :

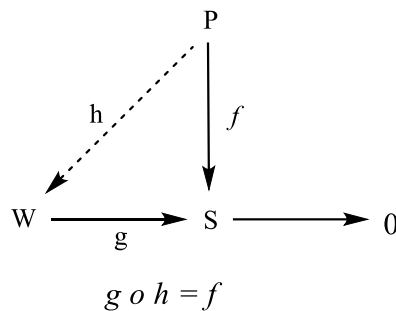


Fig1.  $W$ -Projective module

via homomorphism, with exact row being commutatively extended.  $h: P \rightarrow W$  that is  $goh = f$ .

**Definition 14 :** A sub-module  $S$  of a  $L$  – module  $V$  is called to as fully invariant if,  $f(S) \subseteq S$  for all  $f \in \text{End}_L(V)$ . [4]

**Definition 15:** A  $L$ -module  $V$  is called (co-quasi – D) if  $\text{Hom}(V, S) = 0$  for all proper sub-module  $S$  of  $V$ .

Equivalently "A nonzero  $L$  – module  $V$  is( co-quasi-D) if for each nonzero  $f \in \text{End}V$ ,  $f$  is an epiomorphism[ 10 ] .

**Corollary 2:** Every finitely generated multiplication  $L$ -module  $V$  is Scalar module[ 8 ].

**Definition 16:** A proper sub-module  $S$  of a  $L$ -module  $V$ , is called quasi prime sub-module if whenever  $x, y \in L, m \in V$ , with  $xym \in S$  implies that  $xm \in S$  or  $ym \in S$ . [8]

### 3. NEARLY ENDO QUASI PRIME SUB-MODULES

As generalizations of quasi prime submodules, we introduce the idea of NEndo Quasi Prime Sub-modules in this section and examine some of their characteristics.

**Definition 17:** A proper sub-module  $S$  of a  $L$ -module  $V$  is called Nearly Endo Quasi Prime sub-module of  $V$  (by short NE quasi prime) sub-module if  $(l \circ g)(m) \in S$ , where  $l, h \in \text{End}V, m \in V$ , implies that  $l(m) \in S + J(V)$  or  $h(m) \in S + J(V)$ .

**Example 1:** Consider  $Z_{15}$  as  $Z$ -module,  $S=(\overline{3})$  is NE quasi prime sub-module, since if  $l, g \in \text{End}(Z_{15}), m \in Z_{15}$ ,  $l(x) = 3x, g(x) = x \quad \forall x \in Z_{15}$  and  $J(Z_{15}) = (\overline{3}) \cap (\overline{5}) = (\overline{0})$ ,  $(l \circ g)(3) = l(3) = 9 \in S$ , implies that  $l(3) = 3(3) = 9 \in P + J(Z_{15}) = (\overline{3})$  and  $g(3) = 3 \in S + J(Z_{15}) = (\overline{3})$ .

### 3.1 Remarks and Examples

1. Every End prime is NE quasi prime sub-module.

**Proof:** Let  $S$  be End Prime sub – module of an  $L$ -module  $V$  and

$L, g \in \text{End}(V), m \in V$  such that  $(L \circ g)(m) \in S = L(g(m)) \in S$ , but  $S$  is End Prime sub-module of  $V$ , then  $g(m) \in S$  or  $L(V) \subseteq S$  hence  $g(m) \in S + J(V)$  or  $L(m) \in S + J(V), \forall m \in V$ . Thus  $S$  is NE quasi prime sub – module.

However, in general, the opposite is not true, as in the case of: Take into consideration  $Z_6$  as  $Z$ -module,  $(\overline{2})$  is a sub-module of  $Z_6, (\overline{2})$  is NE quasi prime sub-module, since if  $l, g \in \text{End}(Z_6), l(x) = x + 1$  and  $g(x) = 3, \forall x \in Z_6$ , where  $J(Z_6) = (\overline{2}) \cap (\overline{3}) = (\overline{0})$ ,  $(l \circ g)(1) = l(g(1)) = 4 \in (\overline{2})$  implies that  $l(1) = 2 \in (\overline{2}) + J(Z_6), (l \circ g)(Z_6) \subseteq (\overline{2}) + J(Z_6)$

$$(l \circ g)(Z_6) = \begin{cases} (l \circ g)(0) = 4 \\ (l \circ g)(1) = 4 \\ (l \circ g)(2) = 4 \\ (l \circ g)(3) = 4 \\ (l \circ g)(4) = 4 \\ (l \circ g)(5) = 4 \end{cases} \quad \text{So that } (l \circ g)(Z_6) \subseteq (\overline{2}) + J(Z_6)$$

But it is not End prime sub-module of  $Z_6, l(3) = 4 \in S = (\overline{2})$ , implies that

$3 \notin (\overline{2}) + J(Z_6) = (\overline{2})$  and  $l(Z_6) \not\subseteq (\overline{2}) + J(Z_6) = (\overline{2})$ , where  $l(0) = 1 \notin (\overline{2}) + J(Z_6) = (\overline{2})$ .

2. Let  $P, S$  be two sub – modules of an  $L$  – module  $V$  and  $P \subset S$ . If  $S$  is NE quasi prime sub – module of  $V$ ,

then  $P$  is not necessary that NE quasi prime sub-module of  $V$ , for example: Take into consideration  $Z_{24}$  as  $Z$ -module, Take  $S = (\overline{4}), (\overline{12}), \forall f, g \in \text{End}(Z_{24}), f(x) = x - 2, g(x) = 2x, \forall x \in Z_{24}$  where  $J(Z_{24}) = (\overline{2}) \cap (\overline{3}) = (\overline{6}), S = (\overline{4})$  is NE quasi prime sub-module of  $V = Z_{24}$  since  $(f \circ g)(7) = f(g(7)) = 12 \in S$  implies that  $g(7) = 14 \in S + J(Z_{24}) = (\overline{2})$ , but  $P = (\overline{12})$  is not NE quasi prime sub-module of  $V = Z_{24}$  since

$$(f \circ g)(7) = f(g(7)) = 12 \in P, \text{ then } f(7) = 5 \notin P + J(Z_{24}) = (\overline{6}), \\ g(7) = 14 \notin P + J(Z_{24}) = (\overline{6}).$$

3. Every Endo quasi Prime sub-module of a  $L$  – module  $V$  is NE quasi-prime sub-module.

**Proof:** Let  $S$  be Endo quasi prime sub-module of a  $L$ -module  $V$  and  $f, g \in \text{End}(V)$ ,  $m \in W$  such that  $(f \circ g)(m) \in S$ , but  $S$  is Endo quasi prime sub-module of  $V$ , then  $f(m) \in S$  or  $g(m) \in S$  hence  $f(m) \in S + J(WV)$  or  $g(m) \in S + J(V)$  since  $S \subseteq S + J(V)$ . Thus  $S$  is NE quasi prime sub-module. But the converse of (3) incorrect in general, for example: consider  $Z_{42}$  as  $Z$ -module,

$S = (\overline{8})$  is NE quasi prime sub-module., since if  $f, g \in \text{End}(Z_{24})$ ,  $f(x) = x - 2$ ,  $g(x) = x + 1, \forall x \in Z_{24}$

Where  $J(Z_{24}) = (\overline{2}) \cap (\overline{3}) = (\overline{6})$  such that  $(f \circ g)(9) = f(g(9)) = 8 \in S = (\overline{8})$ , then  $g(9) = 10 \in S + J(Z_{24}) = (\overline{2})$ , but  $S$  is not End Quasi **Prime** sub-module of  $Z_{24}$  since  $f(9) = 7 \notin S$  and  $g(9) = 10 \notin S$ .

4. Let  $P, S$  be two sub-modules of a  $L$ -module  $V$ , and  $P \subset S$ . If  $P$  is NE quasi prime sub-module of  $V$ , then  $P$  is NE quasi prime sub-module of  $S$  with  $J(V) \subseteq J(S)$ .

**Proof:** Let  $(f \circ g)(m) \in P, \forall m \in S$  since  $S < V$ , so  $m \in V, f \in \text{End}(WV)$ , Since  $P$  is NE quasi prime sub-module of  $V$ , then either  $f(m) \in P + J(V)$  or  $g(m) \in P + J(VW)$ , since  $J(V) \subseteq J(S)$ , hence  $f(m) \in P + J(S)$  or  $g(m) \in P + J(S)$ . Thus  $P$  is NE quasi prime sub-module of  $S$ .

5. The intersection of two NE quasi prime sub-module not be NE quasi prime sub-module, for example: consider  $Z_{12}$  as  $Z$ -module take  $P = (\overline{4}), S = (\overline{3})$  is NE quasi prime sub-module of  $Z_{12}$ , since  $\forall f, g \in \text{End}(Z_{12}), f(x) = x - 3, g(x) = x - 2, \forall x \in Z_{12}$  where  $J(Z_{12}) = (\overline{2}) \cap (\overline{3}) = (\overline{6})$  such that  $(f \circ g)(5) = f(g(5)) = 0 \in (\overline{4})$ , then  $f(5) = 2 \in (\overline{4}) + J(Z_{24}) = (\overline{2})$ , also  $(f \circ g)(5) = f(g(5)) = 0 \in (\overline{3})$ , then  $g(5) = 3 \in (\overline{3}) + J(Z_{12}) = (\overline{3})$ . But  $(\overline{4}) \cap (\overline{3}) = (\overline{0})$  is not NE quasi prime sub-module of  $Z_{12}$ , since  $(f \circ g)(5) = f(g(5)) = 0 \in (\overline{0})$ , then  $f(5) = 2 \notin (\overline{0}) + J(Z_{12}) = (\overline{6}), g(5) = 3 \notin (\overline{0}) + J(Z_{12}) = (\overline{6})$ .

6. Every NE quasi prime sub-module is NET-ABSO sub-module.

**Proof:** Let  $S$  be NE quasi prime sub-module of a  $L$ -module  $V$  and  $l, g \in \text{End}(V), m \in V$  such that  $(l \circ g)(m) \in S$ , but  $S$  is NE quasi prime sub-module of  $V$ , then  $l(m) \in S + J(V)$  or  $g(m) \in S + J(V), \forall m \in V$ . Thus  $S$  is NET-ABSO sub-module. However, in general, the opposite is not true, as in the case of: Take into consideration  $Z_{24}$  as  $Z$ -module,  $S = (\overline{8})$  is a sub-module of  $Z_{24}$ , is NET-ABSO sub-module, since if  $l, g \in \text{End}(Z_{24}), l(x) = x + 5$  and  $g(x) = 3, \forall x \in Z_{24}$ , where  $J(Z_{24}) = (\overline{2}) \cap (\overline{3}) = (\overline{6}), (l \circ g)(4) = l(g(4)) = l(3) = 8 \in P = (\overline{8})$  implies that  $l(4) = 9 \notin (\overline{8}) + J(Z_{24}) = (\overline{2})$  and  $g(4) = 3 \notin (\overline{8}) + J(Z_{24}) = (\overline{2})$  or  $(l \circ g)(Z_{24}) \subseteq (\overline{8}) + J(Z_{24}) = (\overline{2})$ ,

$$(\log)(Z_{24}) = \begin{cases} l(g(0)) = 8, & l(g(1)) = 8, \\ l(g(2)) = 8, & l(g(3)) = 8, \\ l(g(4)) = 8, & l(g(5)) = 8, \\ l(g(6)) = 8, & l(g(7)) = 8, \\ l(g(8)) = 8, & l(g(9)) = 8, \\ l(g(10)) = 8, & l(g(11)) = 8, \\ l(g(12)) = 8, & l(g(13)) = 8, \\ l(g(14)) = 8, & l(g(15)) = 8, \dots \end{cases}$$

So that  $(l \circ g)(Z_{24}) \subseteq (\overline{8}) + J(Z_{24}) = (\overline{2})$ .

But it is not NE quasi prime sub-module of  $Z_{24}$ ,  $l(g(4)) = 8 \in S = (\overline{8})$  implies that  $l(4) = 9 \notin (\overline{8}) + J(Z_{24}) = (\overline{2})$  and  $g(4) = 3 \in (\overline{8}) + J(Z_{24}) = (\overline{2})$ .

**Proposition 1:** Let  $S$  be NE quasi prime submodule of an  $L$ -module  $V$  is Scalar module and  $J(V) \subseteq S$  if and only if  $S$  is quasi prime sub-module of  $V$ .

**Proof:**  $(\implies)$  Let  $abm \in S$  for  $a, b \in L, \forall m \in V$  such that  $f(m) = am, g(m) = bm$ , then  $(f \circ g)(m) = abm \in S$ , but  $S$  is NE quasi prime sub-module of  $V$ , then either  $am = f(m) \in S + J(V)$  or  $bm = g(m) \in S + J(V)$ , hence  $f(m) = am \in S$  or  $bm = g(m) \in S$  since  $J(V) \subseteq PS$ . Then  $S$  is **quasi Prime** sub-module of  $V$ .

$(\impliedby)$  Let  $(f \circ g)(m) \in S$  where  $f, g \in \text{End}(V), \forall m \in V$ , since  $V$  is Scalar – module, then there exist  $a, b \in L$  such that  $am = f(m), bm = g(m)$  for each  $m \in V$ , so  $(f \circ g)(m) = abm \in S$ , but  $S$  is **quasi Prime** sub-module of  $V$ , implies that either  $am = f(m) \in S$  or  $bm = g(m) \in S$ , Since  $J(V) \subseteq S$  hence  $f(m) \in S + J(V)$  or  $g(m) \in S + J(V)$  Then  $S$  is NE quasi prime sub-module of  $V$ .

**Remark 1:** In general, the opposite of Proposition 3.5 is not true if the condition of the Scalar-module is removed. The example that follows demonstrates: Let  $Z$ -module  $Z \oplus Z$  and  $S = 3Z \oplus (0)$ , It is evident that  $S$  is quasi Prime sub-module since  $rt(u, w) \in 3Z \oplus (0)$  then  $(rtu, rtw) \in 3Z \oplus (0)$  i.e.  $rtu \in 3Z$  and  $rtw = 0$ . Follows  $w = 0$  and  $ru \in 3Z$  or  $tu \in 3Z$ , so either  $r(u, w) \in 3Z \oplus (0)$  or  $t(u, w) \in 3Z \oplus (0)$ . Thus  $S$  is quasi Prime. But we can such that  $S$  is not E-quasi prime as follows:

Define  $H(u, w) = (w, u), Q(u, w) = (w, 0)$ . It is evident that  $H, Q \in \text{End}V, J(Z)=(0)$ , so  $(Q \circ H)(3,1) = Q(1,3) = (3,0) \in S$ , but  $H(3,1) = (1,3) \notin S + J(Z)$  and  $Q(3,1) = (1,0) \notin S + J(Z)$ .

**Corollary 3:** Let  $S$  be a sub-module of a finitely generated multiplication  $L$ -module  $V$ . Then  $S$  is quasi prime sub-module if and only if  $S$  is NE quasi prime sub-module of  $V$ .

**Proof:**

$(\implies)$  Let  $abm \in S$  for  $a, b \in L, \forall m \in V$  such that  $f(m) = am, g(m) = bm$ , then  $(f \circ g)(m) = abm \in S$ , but  $S$  is NE quasi prime sub-module of  $V$ , then either  $am = f(m) \in S + J(V)$  or  $bm = g(m) \in S + J(V)$ , hence  $f(m) = am \in S$  or  $bm = g(m) \in S$  since  $J(V) \subseteq S$ . Then  $S$  is quasi Prime sub-module of  $V$ .

$(\impliedby)$  Let  $(f \circ g)(m) \in S$  where  $f, g \in \text{End}(V), \forall m \in V$ , since  $V$  is scalar-module, then there exist  $a, b \in L$  such that  $am = f(m), bm = g(m)$  for each  $m \in V$ , so  $(f \circ g)(m) = abm \in S$ , but  $S$  quasi Prime sub-module of  $V$ , implies that either  $am = f(m) \in S$  or  $bm = g(m) \in S$ , since  $J(V) \subseteq S$ , hence  $f(m) \in S + J(V)$  or  $g(m) \in S + J(V)$ . Then  $S$  is NE quasi prime sub-module of  $V$ .

**Proposition 2:** Let  $\varphi: V \rightarrow V$  be an epimorphism,  $S$  is fully invariant Equasi prime sub-module of  $L$ -module  $V$ , such that  $\varphi(V) \not\subseteq S$ . Then  $\varphi^{-1}(S)$  is NE quasi prime sub-module of  $V$ .

**Proof:** It is evident that  $\varphi^{-1}(S)$  a proper sub-module of  $V$ , if not then  $\varphi^{-1}(S) = V$ , hence  $S = \varphi(V) = V$  this contradiction. Let  $l, g \in \text{End}V, u \in V$  such that  $(l \circ g)(u) \in \varphi^{-1}(S)$ , then  $\varphi(l \circ g)(u) \in S$ . since  $S$  is Equasi prime sub-module of  $V$  then either  $\varphi(l(u)) \in S$  or  $\varphi(g(u)) \in S$ , so  $l(u) \in \varphi^{-1}(S)$  or  $g(u) \in \varphi^{-1}(S)$ , that implies  $l(u) \in \varphi^{-1}(S) + J(V)$  or  $g(u) \in \varphi^{-1}(S) + J(V)$ , hence  $l(u) \in \varphi^{-1}(S) + J(V)$  or  $g(u) \in \varphi^{-1}(S) + J(V)$ . Therefore  $\varphi^{-1}(S)$  is NE quasi prime sub-module of  $L$ -module  $V$ .

**Proposition 3:** Let  $V$  as  $L$ -module  $V$  and  $S \subseteq V$ , if  $S$  is NE quasi prime sub-module. Then  $(S :_L(u))$  is prime ideal in  $L, J(V) \subseteq S$ .

**Proof:** Let  $r, t \in (S :_L(u))$ . Then  $ru \in S$  where  $Q, H \in \text{End}V, Q(u) = ru, H(u) = tu$ . So that  $(Q \circ H)(u) = rtu \in S$ , but  $S$  is NE quasi prime sub-module of  $V$ , then  $Q(u) = ru \in S + J(V)$  or  $H(u) = tu \in S + J(V) \Rightarrow ru \in S$  or  $tu \in S$  since  $J(V) \subseteq S$ , hence  $r \in (S :_L(u))$  or  $t \in (S :_L(u))$ .

**Proposition 4:** Let  $S$  be a proper sub-module of duo  $L$ -module  $V$  with  $J(V) \subseteq S$ . Then  $S$  is a NE quasi prime sub-module if and only if  $(S :_{\text{End}V}(u)) = (S :_{\text{End}V} H(u))$  for each  $u \in V, H(u) \notin S, H \in \text{End}V$ .

**Proof:** ( $\Rightarrow$ ) Let  $Q \in (S :_{\text{End}V} H(u)), H(u) \notin S$ , then  $Q(H(u)) \in S$ . but  $S$  is NE quasi prime sub-module, and  $H(u) \notin S$ , So that  $Q(u) \in S + J(V)$ , then  $Q(u) \in S + J(V)$  since  $J(V) \subseteq S$ , thus  $Q \in (S :_{\text{End}V}(u))$ , that is  $(S :_{\text{End}V} H(u)) \subseteq (S :_{\text{End}V}(u))$ . Now let  $Q \in (S :_{\text{End}V}(u))$ , hence  $Q(u) \in S$ , but  $V$  is duo module, so  $H(u) \in (u)$ . it follows that  $Q(H(u)) \subseteq (Q(u)) \subseteq S$ , this implies  $Q \in (S :_{\text{End}V} H(u))$ .

( $\Leftarrow$ ) Let  $Q, H \in \text{End}V$  such that  $(H \circ Q)(u) \in S$  and suppose that  $H(u) \notin S$ .

So  $Q \in (S :_{\text{End}V} H(u))$ . But  $(S :_{\text{End}V} H(u)) = (S :_{\text{End}V}(u))$  by hypothesis, so that  $Q \in (S :_{\text{End}V}(u))$ , hence  $Q(u) \in S$ , so that  $Q(u) \in S + J(V)$ .

Thus  $S$  is an NE quasi prime sub-module.

## Conflicts of Interest

The authors declare no conflicts of interest.

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