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Research Article

Fuzzy Semi and Fuzzy Strongly Semi Two-Absorbing Second Sub-modules

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ABSTRACT

In this work, fuzzy semi Two-Absorbing second sub-modules and fuzzy strongly semi Two-Absorbing second sub-modules are presented. The connections between fuzzy semi prime (fuzzy strongly semi prime) second sub-modules, fuzzy quasi prime (fuzzy strongly quasi prime) second sub-modules, and fuzzy Two-Absorbing (fuzzy strongly Two-Absorbing) second sub-modules are also covered. Here are some fundamental characteristics and attributes of these ideas.

1. INTRODUCTION

Zadeh [16] first put forward fuzzy sets in 1965. Fuzzy groups, a generalization of the prime fuzzy ideal, were first proposed by Rosenfeld [1] in 1971. In 1989, Mukherjee [14] extended this notion to the prime fuzzy ideal. In 1992, Zahedi [17] introduced the idea of a main fuzzy ideal. In 2004, Hadi [4] expanded this idea to include semi-primitive fuzzy sub-modules. The prime fuzzy sub-module notion was first presented by Rabi [14]. The definition of the quasi-prime fuzzy sub-module was first put forth by Hatam in [5] in 2001. In 2017, Deniz S. et al. [2] introduced the idea of the Two-Absorbing fuzzy ideal.

In 2018 and 2019, Hatam and Wafaa introduced the notions of two-absorbing fuzzy sub-modules [6] and semi-two-absorbing fuzzy sub-modules [7]. In the year 2019, H. Ansari Toroghy [3] proposed fuzzy second sub-modules. Wafaa and Assel originally introduced the idea of fuzzy two-absorbing second sub-modules in 2021 [15].

There are three parts to the search. The definition and investigation of the Two-Absorbing fuzzy second sub-module are presented in Section (1). In section (2), along with several hypotheses, theorems, and examples, we define fuzzy semi Two-Absorbing second sub-modules and list the necessary features. The fuzzy strongly semi-two-absorbing second sub-module is examined in section (3). Its properties and relationships with other fuzzy second sub-module concepts are searched for in the remarks and examples.

Note that: The notations fzy sub-module, fzy singleton, and fzy module represent the fuzzy sub-module, fuzzy singleton, and fuzzy module.

2. PRELIMINARIES

Definition 1.1[16]: Let S be a non-empty set and L be an interval [0,1] of the real line (real number). A function from S into L is a fzy set A in S (a fzy subset of S).
Definition 1.2[8]: Let $x_u: S \rightarrow L$ be a fzy set in S, where $x \in S, u \in L$, define by

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$x_u(y) = \begin{cases} u & if\ x = y \\ 0 & if\ x \neq y \end{cases}$, x_u is known as fzy singleton in S, If $x = 0$ and $u = 1$, then $0_1(y) = \begin{cases} 1 & if\ y = 0 \\ 0 & if\ y \neq 0 \end{cases}$

Definition 1.1.8[18]: A fzy ideal of a ring T is a fzy subset K of that ring, if for all $x, y \in T$:

1- $K(x - y) \geq min\{K(x), K(y)\}$.

2- $K(xy) \geq max\{K(x), K(y)\}$.

Definition 1.3[18]: Let W be a T-module fzy set Y of W is called fzy module of a T-module W if
1- $Y(x - y) \geq min\{Y(x), Y(y)\}$, for all $x, y \in W$.
2- $Y(rx) \geq Y(x)$, for all $x \in W, r \in T$.
3- $X(0) = 1$ (0 is the zero element of W).

Definition 1.4[9]: Let Y and A be two fzy modules of a T-module W. A is called fzy sub-module of Y if $A \subseteq Y$.
Proposition 1.5[12]: Let A be fzy set of W. Then the level subset A_u, $\forall u \in L$ is a sub-module of M if and only if A is fzy submodule of fzy module of a T-module W.
Definition 1.6[18]: Let A and B be two fzy sub-modules of fzy module Y. The residual quotient of A and B denoted by $(A{:}B)$ is the fzy subset of T defined by:
$(A{:}B)(r) = sup\{v \in L{:} r_v.B \subseteq A\}$ for all $r \in T$. That $(A{:}B) = \{r_v{:} r_v.B \subseteq A\ ; r_v\ is\ a\ fzy\ singleton\ of\ T\}$. If $B =< x_k >$, then $(A{:}< x_k >) = \{r_v{:} r_v.x_k \subseteq\ A\ ; r_v\ is\ fzy\ singleton\ of\ T\}$.
Definition 1.7 [9]: Assume that A is a proper fzy sub-module of Y. The definition of the fzy annihilator of A, represented by F-annA, is:$(F - annA)(r) = sup\{v{:}\ v\ \in L\ , r_v A \subseteq 0_1\}$, To all $r \in T$.
Note that: $\text{F} - \text{ann}A = (0_1{:}_R\, A)$, hence $(F - annY)_v \subseteq annY_v$, [5].
Proposition 1.8 [9]: F-annY is the fzy ideal of T if Y is the fzy module of a T-module W..
Definition 1.9 [11]: A ring T's prime fzy ideal is its fzy ideal Ĥ, provided that H is non-empty and that for any a_s, b_l fzy singleton of T, $a_s b_l \subseteq$ Ĥ suggests that a choice of $a_s \subseteq$ Ĥ or $b_l \subseteq$ Ĥ, $\forall s, l \in L$.
Definition 1.10 [2]: Let Ĥ be a fzy ideal of R that is not empty. Therefore, Ĥ is referred to as the Two-Absorbing fzy ideal if, for any fzy singletons $a_s,\ b_l,\ r_k$ of T then $a_s b_l\ r_k \subseteq$ Ĥ Suggest that a choice of $a_s b_l \subseteq$ Ĥ or $a_s\ r_k \subseteq$ Ĥ or $b_l\ r_k \subseteq$ Ĥ.
Definition 1.11 [8]: If B is any fzy sub-module of X that contains A, then $B = X$. A maximum fzy sub-module is a correct fzy sub-module A of a fzy module X of a T-module W.
Definition 1.12[3]: Let $A \neq 0_1$ be the fzy second sub-module of a T-module W, and let Y be its fzy module $\forall r \in R$ we've $1_r.A = A$ or $1_r.A = 0_1$ where 1_r is fzy ideal of T.
Definition 1.13[13]: A divisible fuzzy module is a fzy module Y of a T-module W if each fzy singleton $0_1 \neq r_v$ of R such that $r_v \cdot Y = Y$.

Definition 1. 14[15]: Let Y be a T-module's fzy. W. A suitable submodule It is claimed that A of Y is an entirely irreducible fzy sub-module if $A = \bigcap_{i\in I} A_i$,where $\{A_i\}_{i\in I}$ is a family of fzy sub-modules of Y, suggest that $A = A_i$ for any $i \in I$. Every fzy sub-module of Y is an intersection of a fully irreducible fzy sub-module of Y, as is readily apparent.
Definition 1.15[15]: If fzy singleton a_s of T and B are entirely irreducible fzy sub-modules, then let $A \neq 0_1$ be designated as fzy prime (fzy Strongly prime) second sub-module (B be fzy sub-module). $a_s A \subseteq B$, then $A \subseteq B$ or $a_s \subseteq F - ann\ (A)$.
Definition 1.16[15]: Let $A \neq 0_1$ be fzy sub-module of fzy module of Y of a T-module W. A is named fzy Two-Absorbing second sub-module if each time fzy singletons a_s, b_l of T , B is completely irreducible fzy sub-module and $a_s b_l A \subseteq B$ then either $a_s A \subseteq B$ or $b_l A \subseteq B$ or $a_s b_l \subseteq F - ann\ (A)$.
Definition 1.17[15]: Let $A \neq 0_1$ be fzy sub-module of fzy module of Y of a T-module W. A is named fzy quasi-prime (fzy strongly quasi-prime) second sub-module if each time fzy singletons a_s, b_l of R, B is completely irreducible fzy sub-module (B is fzy sub-module) and $a_s b_l A \subseteq B$ then either $a_s A \subseteq B$ or $b_l A \subseteq B$.
Remarks 1.18[15]: The fzy Two-Absorbing second sub-module is clearly the same as every fzy quasi-prime second sub-module.

3. FZY SEMI TWO-ABSORBING SECOND SUB-MODULES

In this part, the fzy semi-two-absorbing second sub-module notion will be examined along with its properties, theorems, assertions, notes, and examples.

Definition 2.1: A non-zero fzy sub-module A of fzy module Y of a T-module W is known as fzy semi prime second sub-module if whenever $a_s^2 A \subseteq K$ where a_s is fzy singleton of T, $s \in L$ and K is completely irreducible fzy sub-module of Y, implies $a_s A \subseteq K$.

Proposition 2.2: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T-module W. Then A is a fzy semi prime second sub-module of Y if and only if the level A_u is semi prime second sub-module of Y_u, for all $u \in$ L.

Proof: $\Longrightarrow$) Let $a^2 A_u \subseteq K_u$ for every $a \in R$ and $A_u \neq 0$ be a submodule of Y_u, K_u be a completely irreducible sub-module of Y_u, we have $aA_u \subseteq K_u$ then $K(aA) \geq u$. So $(a^2 A)_u \subseteq K$ implies that $a_s^2 A_u \subseteq K$. Since A is a fzy semi prime second sub-module, then $a_s A \subseteq K$ equival $a_s^2 A \subseteq K$. Hence, $(aA)_u \subseteq K$ and $(a^2 A)_u \subseteq K$, so that $aA_u \subseteq K_u$ and $a^2 A_u \subseteq K_u$. Thus, A_u is a semi prime second sub-module.

$\Longleftarrow$) Let $a_s^2 A \subseteq K$ for fzy singletons a_s of T ,$\forall s \in L$, hence $(a^2 A)_u \subseteq K$ so that $K(a^2 A) \geq u$ implies $a^2 A \subseteq K_u$, but A_u is a semi prime second sub-module, then $aA_u \subseteq K_u$ and $a^2 A_u \subseteq K_u$. Hence, $(aA)_u \subseteq K$ and $(a^2 A)_u \subseteq K$ implies $a_s A \subseteq K$ and $a_s^2 A \subseteq K$. Thus, A is fzy semi prime second sub-module.

Example 2.3: Let $Y: Z_{10} \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Z_{10} \\ 0 & o.w. \end{cases}$

It is clear Y is a fzy module of Z_{10} as Z-module

Now, $Y_u = Z_{10}$ as Z-module is semi prime second sub-module since $5^2 Y_u \subseteq (\bar{5})$, then $5^2 Y_u = 5Y_u = (\bar{5})$. So that, Y is fzy semi prime second sub-module.

Definition 2.4: A non-zero fzy sub-module A of fzy module Y of a T-module W, is called fzy semi Two-Absorbing second sub-module of Y if whenever $a_s^2 A \subseteq K$ where a_s is fzy singleton of R, $s \in L$ and K is a completely irreducible fzy sub-module, implies either $a_s A \subseteq K$ or $a_s^2 \subseteq F - ann(A)..nn$(es either bmdubmd.) of Y

The following proposition specificities of fzy semi Two-Absorbing second sub-module is given.

Proposition 2.5: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T-module W. Then A is a fzy semi Two-Absorbing second sub-module of Y if and only if the level A_u is a semi Two-Absorbing second sub-module of Y_u, for all $u \in$ L .

Proof: $\Rightarrow$) Let $a^2 A_u \subseteq K_u$ for every $a \in R$ and $A_u \neq 0$ be sub-module of Y_u , K_u be a completely irreducible sub-module of Y_u, $\forall u \in L$, then $K(a^2 A) \geq u$, hence $(a^2 A)_u \subseteq K$ so that, $a_s^2 A_v \subseteq K$ where $u = min\{s, v\}$ and $(a^2)_s = a_s^2$ but A is a fzy semi Two-Absorbing second sub-module, then either $a_s A \subseteq K$ or $a_s^2 \subseteq F - ann(A)$. Hence,$(aA)_u \subseteq K_u$ or $(a^2)_u \subseteq ann(A_u)$, implies $aA_u \subseteq K_u$ or $a^2 \in ann(A_u)$. Thus, A_u is a semi Two-Absorbing second sub-module of Y_u.

$\Leftarrow$) Let $a_s^2 A \subseteq K$, a_s is a fzy singleton of T and K be completely irreducible fzy sub-module of Y, then $(a^2 A)_u \subseteq K$ where $u = min\{s, 1\}$, hence $K(a^2 A) \geq u$ so that, $a^2 A_u \leq K_u$. But A_u is semi Two-Absorbing second sub-module of Y_u, then either $aA_u \subseteq K_u$ or $a^2 \in ann(A_u)$, hence $(aA)_u \subseteq K$ or $(a^2)_u \subseteq ann(A_u)$, so that $a_s A \subseteq K$ or $a_s^2 \subseteq F - ann(A)$. Thus, A is a fzy semi Two-Absorbing second sub-module of Y.

Remarks and Examples 2.6:

1- In general, the opposite is not true; for instance, every fzy-semi prime second sub-module is fzy-semi Two-Absorbing second sub-module:

Let $Y: Z_9 \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Z_9 \\ 0 & o.w. \end{cases}$

It is clear Y is a fzy module of Z_9 as Z-module

Now, $Y_u = Z_9$ as Z-module is a semi Two-Absorbing second sub-module since $3^2 Y_u = 0$ but it is not semi prime second sub-module since $3^2 Y_u = (0)$ but $3Y_u \neq (0)$ so that, Y is fzy semi Two-Absorbing second sub-module, but it is not fzy-semi prime second sub-module

2- Every fzy Two-Absorbing second sub-module is fzy semi Two-Absorbing second sub-module. However, In general, the opposite is untrue. For instance:

Let $Y: \mathrm{Z}_{20} \to \mathrm{L}$ where $Y(y) = \begin{cases} 1 & \text{if } \mathrm{y} \in \mathrm{Z}_{20} \\ 0 & \mathrm{o.w.} \end{cases}$

As a Z-module, it is clear that Y is a fzy module of Z_{20}.

Now, $\mathrm{Y_u} = \mathrm{Z}_{20}$ as Z- module, for all $u \in L$, is semi Two-Absorbing second sub-module since $5^2(Y_u) \subseteq (\bar{5})$, then $5(Y_u) \subseteq (\bar{5})$, but it is not Two-Absorbing second sub-module since $2.5(Y_u) \subseteq (10)$, but $2.(Y_u) \nsubseteq (10)$, $5.(Y_u) \nsubseteq (10)$ and $2.5 \notin ann(Y_u) = 20Z$. So that, Y is a fzy- semi Two-Absorbing second sub-module, which it is not fzy Two-Absorbing second sub-module of Y.

3- Every fzy quasi prime second sub-module is a fzy semi Two-Absorbing second sub-module, However, as demonstrated by the example in part (2), the opposite is generally false.

Proposition 2.7: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T-module W. Then the following expressions are equivalent:

1- A is fzy semi Two-Absorbing second sub-module and F-ann(A) is a fzy semi prime ideal.

2- A is a fzy prime second sub-module.
3- A is a fzy semi prime second sub-module.
4- A is a fzy quasi prime second sub-module.
5- A is fzy Two-Absorbing second sub-module and F-ann(A) is a fzy prime ideal.

Proof: (1) → (2)
Let $a_s(a_sA) \subseteq K$ for fzy singleton a_s of R. Since A is fzy semi Two-Absorbing second sub-module, then $a_sA \subseteq K$ or $a_s^2 \subseteq F - ann(A)$. If $a_sA \subseteq K$ then we are done. If $a_s^2 \subseteq F - ann(A)$ then $a_s \subseteq F - ann(A)$ since $F - ann(A)$ is a fzy semi prime ideal. So that, A is a fzy prime second sub-module.
(1) → (3)
Let $a_s^2A \subseteq K$ for fzy singleton a_s of T. Since A is fzy Two-Absorbing second sub-module, then $a_sA \subseteq K$ or $a_s^2 \subseteq F - ann(A)$, If $a_sA \subseteq K$ the proof is complete. If $a_s^2 \subseteq F - ann(A)$, then $a_s \subseteq F - ann(A)$ since F-ann(A) is a fzy semi prime ideal. Hence, $a_sA \subseteq K$. Thus, A is a fzy semi prime second sub-module.
(2) → (3)
Let $a_s^2A \subseteq K$ for fzy singelton a_s of T. Since A is a fzy prime second sub-module, then $a_sA \subseteq K$ then we are done .
(3) → (4) Let $a_s(a_sA) \subseteq K$ for fzy singleton a_s of T. Since A is a fzy semi prime second submodule, then $a_sA \subseteq K$, shows that A is a second submodule of fzy quasi prime.
(4) → (5) Since A is a fzy quasi-prime second submodule, then A is a fzy Two-Absorbing second sub-module by Remarks (1.18)
Now , let $a_sb_l \subseteq F - ann(A)$, for fzy singletons a_s , b_l of R then $a_sb_lA \subseteq 0_1$, hence $a_sA \subseteq 0_1$ or $b_lA \subseteq 0_1$. Thus, $a_s \subseteq F - ann(A)$ or $b_l \subseteq F - ann(A)$, so that F-ann(A) is a fzy prime ideal.
(5) → (1) It is clear.

4. FZY STRONGLY SEMI TWO-ABSORBING SECOND SUBMODULE

In this section, the definition of the strongly semi Two-Absorbing fzy second sub-module is discussed, along with some of the associated conclusions and proofs.

Definition 3.1: A non-zero fzy submodule A of fzy module Y of a T-module W, is called a fzy strongly semi Two-Absorbing second submodule if whenever $a_s^2A \subseteq K$ enever led strongly semi T-ABSO F.second subm, where a_s fzy singleton of T and K fzy sub-module of Y, implies either $a_sA \subseteq K$ or $a_s^2 \subseteq F - annA$ Equivalently if $A \neq 0_1$ fzy sub-module is called a fzy strongly semi Two-Absorbing second submodule $a_s^2A = a_sA$ or $a_s^2A = 0_1$ for all a_s fzy singleton of R.
Proposition 3.2: Let $A \neq 0_1$ be a fzy sub-module of fzy module Y of a T-module. Then A is a fzy strongly semi Two-Absorbing second sub-module of Y if and only if the level A_u is a strongly semi Two-Absorbing second sub-module of Y_u, for all $u \in L$.
Proof: The same proof of Proposition (2.5) only replaces K is a completely irreducible fzy sub-module by K is a fzy submodule of Y.
Proposition 3.3 : Let Y be a fzy module of a T-module M. A non-zero fzy sub-module A of Y is a fzy strongly semi Two-Absorbing second submodule of Y if and only if $(K:_R A)$ is a fzy semi prime ideal of T for each fzy sub-module $A \nsubseteq K$ in Y and $a_s^2A \neq 0_1$ for each fzy singleton a_s of T .

Proof:
⇒) Let $A \neq 0_1$ be a fzy strongly semi Two-Absorbing second submodule of Y and K fzy sub-module Y such that $A \nsubseteq K$ implies $(K: A)$ be a proper fzy ideal of T. Let a_s be fzy singleton of T such that $a_s^2 \in (K: A)$, then $a_s^2A \subseteq K$ and $a_s^2A \neq 0_1$, then $a_sA \subseteq K$. Hence, $a_s \in (K: A)$. Thus, $(K: A)$ is a fzy semi prime ideal of T.
⇐) Let K be a fzy sub-module of Y and $(K: A)$ be fzy semi prime, let a_s be fzy singleton of T such that $a_s^2A \subseteq K$ ingleton of R such that. In case $A \subseteq K$, then $a_sA \subseteq K$. If $A \nsubseteq K$ and $a_s^2 \in (K: A)$, then $a_s \in (K: A)$ since $(K: A)$ is a fzy semi prime ideal, so that $a_sA \subseteq K$. Thus, A is a fzy strongly semi Two-Absorbing second sub-module.
Corollary 3.4 : If A is a fzy strongly semi Two-Absorbing second sub-module, then F-ann(A) is a fzy semi prime ideal of T.

Proof: By Proposition (3.3), it is complete proof.
Remark 3.5: In general, the opposite of Corollary (3.4) is not true, as demonstrated by:
Let $Y: Z \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$
Since Z is a Z-module, it is clear that Y is a fzy module of Z..
Let $A: Z \to L$ where $A(y) = \begin{cases} u & if\ y \in PZ \\ 0 & o.w. \end{cases}$

P is a prime number in this case. A's status as a fzy submodule of Y is clear.
Now, $A_u = PZ$ and $Y_u = Z$ as Z-module, $ann(A_u) = 0$ is a semi prime ideal of Z, but A_u is not strongly semi Two-Absorbing second sub-module since $P^2 A_u \subseteq P^2 A_u$ but $PA_u \nsubseteq P^2 A_u$ and $P^2 \notin ann(A_u) = 0$. So that $F - ann(A) = 0_1$ is a semi prime fzy ideal, but A is not fzy strongly semi Two-Absorbing second sub-module of Y.

Proposition3.6: Anon zero fzy sub-module A of fzy module Y of a T-module W is a fzy strongly semi Two-Absorbing second sub-module if and only if for each fzy ideal H of t and for fzy sub-module K of Y such that $H^2 A \subseteq K$ and $H^2 A \neq 0_1$ implies H$A \subseteq K$.

Proof:

⇒) Let A be a fzy strongly semi Two-Absorbing second submodule of Y. Then $A \neq 0_1$, let H be a fzy ideal and K fzy submodule of Y. If $A \nsubseteq K$, then either $H^2 A \nsubseteq K$ and so nothing to prove or $H^2 A \subseteq K$, hence $H^2 \subseteq (K:_R A)$ and by Proposition (3.3) is fzy semi prim ideal of R, so we have $H \subseteq (K:_R A)$, then $HA \subseteq K$.

⇐) It is clear.

Remarks and Examples 3.7:

1- Every fzy strongly semi prime second sub-module is fzy strongly semi Two-Absorbing second submodule, However, generally speaking, the opposite is not true. For instance:
Let $Y: Z_4 \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Z_4 \\ 0 & o.w. \end{cases}$
It is clear Y is a fzy module of Z_4 as Z-module
Now, $Y_u = Z_4$ as Z-module is strongly semi Two-Absorbing second sub-module since $2^2 Y_u = 0$ but it is not strongly semi prime second sub-module since $2^2 Y_u = (0)$ but $2Y \neq (0)$ so that, Y is a fzy strongly semi Two-Absorbing second sub-module, but it is not strongly semi prim fzy second sub-module.

2- It is evident that the second submodule that is fzy strongly quasi-prime is also fzy strongly semi Two-Absorbing; however, this is not always the case. For instance:
Let $Y: Z_6 \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Z_6 \\ 0 & o.w. \end{cases}$
It is clear that Y is fzy module of Z_6 as Z-module
Let $A: Z_6 \to L$ where $Y(y) = \begin{cases} \frac{1}{2} & if\ y \in Z_6 \\ 0 & o.w. \end{cases}$
It is clear A is a fzy sub-module of Y.
A is a fzy strongly semi Two-Absorbing second sub-module since $a_s^2 A = a_s A$ for each fzy singleton of Z, However, the second sub-module is not a fzy strongly quasi-prime one since $2_{\frac{1}{3}}.3_{\frac{1}{3}}\ A = 0_{\frac{1}{3}} \subseteq 0_1$ but $2_{\frac{1}{3}}\ A \nsubseteq 0_1$ and $3_{\frac{1}{3}}\ A \nsubseteq 0_1$.

3- Every fzy second sub-module is a fzy strongly semi Two-Absorbing second sub-module, but the converse incorrect in general, for example, see the example in part (2) where A is a fzy strongly semi Two-Absorbing second sub-module but it is not fzy second submodule since $a_s A \neq A$ for each a_s fzy singleton of Z.

4- A fzy secondary sub-module is a fzy strongly semi Two-Absorbing second sub-module. The conversely is not true, see example part (2) A is a fzy strongly semi Two-Absorbing second sub-module, but it is not a fzy secondary sub-module since $a_s A \neq A$ and $a_s^2 A \neq 0_1$.

5- It is obvious that a fzy strongly Two-Absorbing second sub-module is a fzy strongly semi Two-Absorbing second sub-module. In general, the opposite is not true. For instance:
Let $Y: Z_6 \oplus Z_{p^\infty} \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Z_6 \oplus Z_{p^\infty} \\ 0 & o.w. \end{cases}$
Where p is a prime number, it is clear that Y is a fzy module of $Z_6 \oplus Z_{p^\infty}$ as Z- module. Now, $Y_u = Z_6 \oplus Z_{p^\infty}$ as Z-module is strongly semi Two-Absorbing second submodule since $a^2 Y_u = aY_u$ for each $a \in Z$, but it is not strongly Two-Absorbing second submodule since $2.3Y_u = 0 \oplus Z_{p^\infty}$ we have neither $2Y_u \subseteq 0 \oplus Z_{p^\infty}$, nor $3Y_u \subseteq 0 \oplus Z_{p^\infty}$ and nor $2.3Y_u \subseteq (0)Y_u$. So that, Y is a fzy strongly semi Two-Absorbing second submodule, but it is not fzy strongly Two-Absorbing second sub-module.

6- If A is maximal fzy sub-module and hence (prime fzy sub-module) then A may not be fzy strongly semi Two-Absorbing second sub-module for example:
Let $Y: Z \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Z \\ 0 & o.w. \end{cases}$
It is clear Y is a fzy module of Z as Z- module.
Let $A: Z \to L$ where $Y(y) = \begin{cases} \frac{1}{2} & if\ y \in PZ \\ 0 & o.w. \end{cases}$

Where p is a prime number, it is clear A is a fzy submodule of Y.
A is a maximal fzy submodule of Y, but A is not strongly semi Two-Absorbing fzy second submodule since $a_s^2 A \neq a_s A$ and $a_s^2 A \neq 0_1$ for each $a_s \neq 0_1$ fzy singleton of Z.

7- Let A and B be fzy submodules of fzy module Y of a T-module W, with $A \subseteq B \subseteq Y$. If B is a fzy strongly semi Two-Absorbing second sub-modules then A may not be a fzy strongly semi Two-Absorbing second sub-module of Y, for example:

Let $Y: Q \to L$ where $Y(y) = \begin{cases} 1 & if\ y \in Q \\ 0 & o.w. \end{cases}$

It is clear that Y is a fzy module of Q as Z-module.

Let $A: Q \to L$ where $A(y) = \begin{cases} \frac{1}{2} & if\ y \in Z \\ 0 & o.w. \end{cases}$

It is clear that A is a fzy submodule of Y

Let $B = Y$ where $A \subseteq B \subseteq Y$, since B is divisible fzy module if X is a fzy divisible sub-module of itself.

Then B is fzy strongly semi Two-Absorbing second sub-module but the fzy sub-module A is not fzy strongly semi Two-Absorbing second sub-module since $a_s^2 A \neq a_s A$ and $a_s^2 A \neq 0_1$ for each $a_s \neq 0_1$ fzy singleton of Z.

Conflicts Of Interest

The author's paper declares that there are no relationships or affiliations that could create conflicts of interest.

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