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Research Article

# Outcome for the Partition (9,6,3)

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**ABSTRACT**

The set of all irreducible polynomial representations of general linear group $\mathrm{GL}_n(\mathcal{F})$ of degree $n$ is described by the module $\{\mathcal{L}_\lambda(\mathcal{F})\}$; where $\lambda$ runs over all partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_s)$. There are a number of classical formulas that express the formal character of the representation $\mathcal{L}_\lambda(\mathcal{F})$ in terms of standard symmetric polynomials. Such formulas are also valid for the more general representation modules $\{\mathcal{L}_{\lambda/\mu}(\mathcal{F})\}$ associated to skew partition $\lambda/\mu$; where $\mu\subseteq\lambda$.
Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{F}$ a free $\mathcal{R}$-module. The Weyl module resolution studied by Buchsbaum where the Weyl module $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ is the image of the Weyl map $d'_{\lambda/\mu}(\mathcal{F})$ for the skew-partition $\lambda/\mu$.
Reduction the terms of the resolution of the characteristic-free of Weyl module to the terms of the resolution of Lascoux by employing the boundary maps for the partition (9,6,3) and prove that the sequence of the reduction terms is exact.

## 1. INTRODUCTION

The precise definitions of the boundary maps are given in [1]; where it is proved that the complex resolution $\mathcal{B}_\bullet$ in characteristic-zero of $\mathcal{L}_\lambda(F)$ is exact, where

$$\mathcal{B}_\bullet: 0\longrightarrow \mathcal{B}_{\binom{k}{2}} \xrightarrow{\partial_{\binom{k}{2}}} \ldots \longrightarrow \mathcal{B}_1 \xrightarrow{\partial_1} \mathcal{B}_0 \longrightarrow \mathcal{L}_{\lambda/\mu}(\mathcal{F}) \longrightarrow 0$$

Note that the terms of the resolution $\mathcal{B}_\bullet$ of $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$ are direct sums of tensor products of the fundamental representations of $GL_n(\mathcal{F})$.
Hassan generalized the techniques in [2] for the partitions (3,3,3), and (4,4,3) in [3,4] respectively, also authors in [5-7] studied the cases (8,7,3), (6,6,4;0,0), (7,7,4;0,0).
The reduction resolution terms of Weyl module from characteristic-free to Lascoux found in this work and prove that the sequence of these terms is exact.

## 2. THE TERMS OF CHARACTERISTIC-FREE RESOLUTION

We stratify the following formula for the case of partition $(p,q,r)$ to obtain the terms of the resolution for the partition (9,6,3), [2].

$$Res([p,q;0])\otimes\mathcal{D}_r\oplus\sum_{e\geq0}\mathcal{Z}_{32}^{(e+1)}y\,Res([p,q+e+1;e+1])\otimes\mathcal{D}_{r-e-1}\oplus$$
$$\sum_{e_1\geq0,e_2\geq e_1}\mathcal{Z}_{32}^{(e_2+1)}y\mathcal{Z}_{31}^{(e_1+1)}zRes([p+e_1+1,q+e_2+1;e_2-e_1])\otimes\mathcal{D}_{r-(e_1+e_2+2)};$$

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where $\underline{\mathcal{Z}}_{ab}^{(m)}$ is the pursue Bar complex:

$$0 \to \underbrace{\mathcal{Z}_{ab}w\mathcal{Z}_{ab}w \dots \mathcal{Z}_{ab}}_{m-times} \to \sum_{k_i \geq 1, \sum k_i = m} \mathcal{Z}_{ab}^{(k_1)} w \mathcal{Z}_{ab}^{(k_2)} w \dots \mathcal{Z}_{ab}^{(k_{m-1})} \to \cdots \to \mathcal{Z}_{ab}^{(m)} \to 0.$$

Hence the terms of the resolution for the partition (9,6,3) is

$$Res([9,6;0]) \otimes \mathcal{D}_3 \oplus \sum_{e \geq 0} \underline{\mathcal{Z}}_{32}^{(e+1)} y\, Res([9,6+e+1;e+1]) \otimes \mathcal{D}_{3-e-1} \oplus \sum_{e_1 \geq 0, e_2 \geq e_1} \underline{\mathcal{Z}}_{32}^{(e_2+1)} y \underline{\mathcal{Z}}_{31}^{(e_1+1)} z Res([9+e_1+1, 6+e_2+1; e_2-e_1]) \otimes \mathcal{D}_{3-(e_1+e_2+2)} \quad (1)$$

So

$$\sum_{e \geq 0} \underline{\mathcal{Z}}_{32}^{(e+1)} y\, Res([9,6+e+1;e+1]) \otimes \mathcal{D}_{3-e-1} = \underline{\mathcal{Z}}_{32} y\, Res([9,7;1]) \otimes \mathcal{D}_2 \oplus \underline{\mathcal{Z}}_{32}^{(2)} y\, Res([9,8;2]) \otimes \mathcal{D}_1 \oplus \underline{\mathcal{Z}}_{32}^{(3)} y\, Res([9,9;3]) \otimes \mathcal{D}_0,$$

and

$$\sum_{e_1 \geq 0, e_2 \geq e_1} \underline{\mathcal{Z}}_{32}^{(e_2+1)} y \underline{\mathcal{Z}}_{31}^{(e_1+1)} z Res([9+e_1+1, 6+e_2+1; e_2-e_1]) \otimes \mathcal{D}_{3-(e_1+e_2+2)} = \underline{\mathcal{Z}}_{32} y \underline{\mathcal{Z}}_{31} z Res([10,7;0]) \otimes \mathcal{D}_1 \oplus \underline{\mathcal{Z}}_{32}^{(2)} y \underline{\mathcal{Z}}_{31} z Res([10,8;1]) \otimes \mathcal{D}_0;$$

where $\underline{\mathcal{Z}}_{32} y$ is the Bar complex:

$$0 \longrightarrow \mathcal{Z}_{32} y \xrightarrow{\partial_y} \mathcal{Z}_{32} \longrightarrow 0,$$

$\underline{\mathcal{Z}}_{32}^{(2)} y$ is the Bar complex:

$$0 \longrightarrow \mathcal{Z}_{32} y \mathcal{Z}_{32} y \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(2)} y \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(2)} \longrightarrow 0,$$

$\underline{\mathcal{Z}}_{32}^{(3)} y$ is the Bar complex:

$$0 \longrightarrow \mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y \xrightarrow{\partial_y} \begin{matrix} \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \\ \oplus \\ \mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \end{matrix} \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(3)} y \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(3)} \longrightarrow 0,$$

and $\underline{\mathcal{Z}}_{31} z$ is the Bar complex:

$$0 \longrightarrow \mathcal{Z}_{31} z \xrightarrow{\partial_z} \mathcal{Z}_{31} \longrightarrow 0;$$

where $x$, $y$ and $z$ stand for the separator variables, and the boundary map is $\partial_x + \partial_y + \partial_z$.

Let Bar $(\mathcal{M}, \mathcal{A}; \mathcal{S})$ be the free Bar module on the set $\mathcal{S} = \{x, y, z\}$; where $\mathcal{A}$ is the free associative algebra generated by $\mathcal{Z}_{21}$, $\mathcal{Z}_{32}$, and $\mathcal{Z}_{31}$ and their divided powers with the following relations:

$$\mathcal{Z}_{32}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{32}^{(a)} \quad \text{and} \quad \mathcal{Z}_{21}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{21}^{(a)}.$$

And the module $\mathcal{M}$ is the direct sum of $\mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r$ for suitable $p, q$, and $r$ with the action of $\mathcal{Z}_{21}$, $\mathcal{Z}_{32}$, and $\mathcal{Z}_{31}$ and their divided powers.

The terms of the characteristic-free resolution (4.3.1); where $b, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2 \in \mathbb{Z}^+$ are:

∘ In dimension zero $(\mathcal{X}_0)$ we have $\mathcal{D}_9 \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$.

∘ In dimension one $(\mathcal{X}_1)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{6-b} \otimes \mathcal{D}_3$; where $1 \leq b \leq 6$.
- $\mathcal{Z}_{32}^{(b)} y \mathcal{D}_9 \otimes \mathcal{D}_{6+b} \otimes \mathcal{D}_{3-b}$; where $1 \leq b \leq 3$.

∘ In dimension two $(\mathcal{X}_2)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|} \otimes \mathcal{D}_3$; where $2 \leq |b| = b_1 + b_2 \leq 6$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{7-b} \otimes \mathcal{D}_2$; where $2 \leq b \leq 7$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{8-b} \otimes \mathcal{D}_1$; where $3 \leq b \leq 8$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{9-b} \otimes \mathcal{D}_0$; where $4 \leq b \leq 9$.
- $\mathcal{Z}_{32}^{(b_1)} y \mathcal{Z}_{32}^{(b_2)} y \mathcal{D}_9 \otimes \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{3-|b|}$; where $2 \leq |b| = b_1 + b_2 \leq 3$.
- $\mathcal{Z}_{32}^{(b)} y \mathcal{Z}_{31} z \mathcal{D}_{10} \otimes \mathcal{D}_{8+b} \otimes \mathcal{D}_{2-b}$; where $1 \leq b \leq 2$.

∘ In dimension three $(\mathcal{X}_3)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|} \otimes \mathcal{D}_3$; where $3 \le |b| = \sum_{i=1}^{3} b_i \le 6$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_2$; where $3 \le |b| = b_1 + b_2 \le 7$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where $4 \le |b| = b_1 + b_2 \le 8$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{8-b} \otimes \mathcal{D}_1$; where $3 \le b \le 8$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $5 \le |b| = b_1 + b_2 \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)} y \mathcal{Z}_{32}^{(c_2)} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{9-b} \otimes \mathcal{D}_0$; where $c_1 + c_2 = 3$ and $4 \le b \le 9$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{10+b} \otimes \mathcal{D}_{7-b} \otimes \mathcal{D}_1$; where $1 \le b \le 7$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{10+b} \otimes \mathcal{D}_{8-b} \otimes \mathcal{D}_0$; where $2 \le b \le 8$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{D}_{10} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$.

∘ In dimension four $(\mathcal{X}_4)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|} \otimes \mathcal{D}_3$;where $4 \le |b| = \sum_{i=1}^{4} b_i \le 6$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_2$; where $4 \le |b| = \sum_{i=1}^{3} b_i \le 7$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where $5 \le |b| = \sum_{i=1}^{3} b_i \le 8$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where $4 \le |b| = b_1 + b_2 \le 8$; and $b_1 \ge 3$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $6 \le |b| = \sum_{i=1}^{3} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)} y \mathcal{Z}_{32}^{(c_2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $c_1 + c_2 = 3$, $5 \le |b| = b_1 + b_2 \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{9+b} \otimes \mathcal{D}_{9-b} \otimes \mathcal{D}_0$; where $4 \le b \le 9$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_1$; where $2 \le |b| = b_1 + b_2 \le 7$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_0$; where $3 \le |b| = b_1 + b_2 \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{10+b} \otimes \mathcal{D}_{8-b} \otimes \mathcal{D}_0$; where $2 \le b \le 8$.

∘ In dimension five $(\mathcal{X}_5)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|} \otimes \mathcal{D}_3$ ; where $5 \le |b| = \sum_{i=1}^{5} b_i \le 6$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_2$; where $5 \le |b| = \sum_{i=1}^{4} b_i \le 7$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where $6 \le |b| = \sum_{i=1}^{4} b_i \le 8$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where $5 \le |b| = \sum_{i=1}^{3} b_i \le 8$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $7 \le |b| = \sum_{i=1}^{4} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)} y \mathcal{Z}_{32}^{(c_2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $c_1 + c_2 = 3$, $6 \le |b| = \sum_{i=1}^{3} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $5 \le |b| = b_1 + b_2 \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_1$; where $3 \le |b| = \sum_{i=1}^{3} b_i \le 7$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_0$; where $4 \le |b| = \sum_{i=1}^{3} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_0$; where $3 \le |b| = b_1 + b_2 \le 8$ and $b_1 \ge 2$.

∘ In dimension six $(\mathcal{X}_6)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{6-|b|} \otimes \mathcal{D}_3$; where $6 \le |b| = \sum_{i=1}^{6} b_i \le 6$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32} y\, \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_2$; where $6 \le |b| = \sum_{i=1}^{5} b_i \le 7$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)} y\, \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where $7 \le |b| = \sum_{i=1}^{5} b_i \le 8$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where $6 \le |b| = \sum_{i=1}^{4} b_i \le 8$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32}^{(3)} y\, \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $8 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)} y \mathcal{Z}_{32}^{(c_2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $c_1 + c_2 = 3$, $7 \le |b| = \sum_{i=1}^{4} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where $6 \le |b| = \sum_{i=1}^{3} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_1$; where $4 \le |b| = \sum_{i=1}^{4} b_i \le 7$ and $b_1 \ge 1$.

- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_0$; where $5 \le |b| = \sum_{i=1}^{4} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_0$; where $4 \le |b| = \sum_{i=1}^{3} b_i \le 8$ and $b_1 \ge 2$.

∘ In dimension seven $(\mathcal{X}_7)$ we have the sum of the following terms:

- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{16}\otimes\mathcal{D}_0\otimes\mathcal{D}_2$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_1$; where $7 \le |b| = \sum_{i=1}^{5} b_i \le 8$ and e $b_1 \ge 3$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(3)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $7 \le |b| = \sum_{i=1}^{4} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)}\mathcal{y}\mathcal{Z}_{32}^{(c_2)}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3$, $8 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(4)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}.\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{7-|b|}\otimes\mathcal{D}_1$; where $5 \le |b| = \sum_{i=1}^{5} b_i \le 7$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_0$; where $5 \le |b| = \sum_{i=1}^{4} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_0$; where $6 \le |b| = \sum_{i=1}^{5} b_i \le 8$ and $b_1 \ge 2$.

∘ In dimension eight $(\mathcal{X}_8)$ we have the sum of the following terms:

- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(3)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $8 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)}\mathcal{y}\mathcal{Z}_{32}^{(c_2)}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{Z}_{21}^{(b_6)}\mathcal{x}\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3$, $9 \le |b| = \sum_{i=1}^{6} b_i \le 9$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{Z}_{21}^{(b_6)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{7-|b|}\otimes\mathcal{D}_1$; where $6 \le |b| = \sum_{i=1}^{6} b_i \le 7$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_0$; where $6 \le |b| = \sum_{i=1}^{5} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{Z}_{21}^{(b_6)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_0$; where $7 \le |b| = \sum_{i=1}^{6} b_i \le 8$ and $b_1 \ge 2$.

∘ In dimension nine $(\mathcal{X}_9)$ we have the sum of the following terms:

- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(4)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{Z}_{21}^{(b_5)}\mathcal{x}\mathcal{Z}_{21}^{(b_6)}\mathcal{x}\mathcal{D}_{10+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_0$ where $7 \le |b| = \sum_{i=1}^{5} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(2)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$.

∘ Finally, in dimension ten $(\mathcal{X}_{10})$ we have:

- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(2)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$.

## 3. THE LASCOUX RESOLUTION

The terms of the Lascoux complex are obtained by the determinantal expansion of the Jacobi-Trudi matrix of the partition [1]. The positions of the terms of the complex are determined by the length of the permutation to which they correspond, [2].

In the case of the partition (9,6,3) we get the following matrix:

$$\begin{bmatrix} \mathcal{D}_9\mathcal{F} & \mathcal{D}_5\mathcal{F} & \mathcal{D}_1\mathcal{F} \\ \mathcal{D}_{10}\mathcal{F} & \mathcal{D}_6\mathcal{F} & \mathcal{D}_2\mathcal{F} \\ \mathcal{D}_{11}\mathcal{F} & \mathcal{D}_7\mathcal{F} & \mathcal{D}_3\mathcal{F} \end{bmatrix}$$

Then the Lascoux complex has the correspondence between its terms as pursues:

$\mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \leftrightarrow identity.$

$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \leftrightarrow (12).$

$\mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \leftrightarrow (23).$

$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \leftrightarrow (123).$
$\mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \leftrightarrow (132).$
$\mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \leftrightarrow (13).$

Thus the resolution of Lascoux in the case of the partition (9,6,3) has the formulation:

$$\mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \longrightarrow \begin{matrix} \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \\ \oplus \\ \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \end{matrix} \longrightarrow \begin{matrix} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{matrix} \longrightarrow \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$$

## 4. THE OUTCOME

As in [2], we exhibit the terms of the complex (2.1) as:
$\mathcal{X}_0 = \mathcal{L}_0 = \mathcal{M}_0$
$\mathcal{X}_1 = \mathcal{L}_1 \oplus \mathcal{M}_1$
$\mathcal{X}_2 = \mathcal{L}_2 \oplus \mathcal{M}_2$
$\mathcal{X}_3 = \mathcal{L}_3 \oplus \mathcal{M}_3$
$\mathcal{X}_\mathrm{j} = \mathcal{M}_\mathrm{j}$ ; for $\mathrm{j} = 4,5,\dots,10,$
where $\mathcal{L}_e$ are the sum of the Lascoux terms and $\mathcal{M}_e$ are the sum of the others.

Now, we define the map $\sigma_1: \mathcal{M}_1 \longrightarrow \mathcal{L}_1$ such that
$\delta_{\mathcal{L}_1\mathcal{L}_0} \circ \sigma_1 = \delta_{\mathcal{M}_1\mathcal{M}_0}$ …(4.1)
As follows:

- $\mathcal{Z}_{21}^{(2)}x(v) \longmapsto \frac{1}{2}\,\mathcal{Z}_{21}x\partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x(v) \longmapsto \frac{1}{3}\,\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(4)}x(v) \longmapsto \frac{1}{4}\,\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(5)}x(v) \longmapsto \frac{1}{5}\,\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(6)}x(v) \longmapsto \frac{1}{6}\,\mathcal{Z}_{21}x\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{32}^{(2)}y(v) \longmapsto \frac{1}{2}\,\mathcal{Z}_{32}y\partial_{32}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_8 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(3)}y(v) \longmapsto \frac{1}{3}\,\mathcal{Z}_{32}y\partial_{32}^{(2)}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$

It is clear that $\sigma_1$ satisfies (4. 1), then we can define:

$$\partial_1: \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \quad \text{as} \quad \partial_1 = \delta_{\mathcal{L}_1\mathcal{L}_0}$$

At this point, we are in a position to define:

$$\partial_2: \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \quad \text{by} \quad \partial_2 = \delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}$$

**Lemma (4.1):**
The composition $\partial_1\partial_2$ equal to zero.
**Proof:**

$$\partial_1\partial_2(a) = \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \left(\delta_{\mathcal{L}_2\mathcal{L}_1}(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a)\right)$$
$$= \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{L}_1\mathcal{L}_0} \circ (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a).$$

But $\delta_{\mathcal{L}_1\mathcal{L}_0} \circ \sigma_1 = \delta_{\mathcal{M}_1\mathcal{M}_0}$ then we get:
$\partial_1\partial_2(a) = \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{M}_1\mathcal{M}_0} \circ \delta_{\mathcal{L}_2\mathcal{M}_1}(a).$
By properties of the boundary map $\delta$ we get $\partial_1\partial_2 = 0$

We need to define the map $\sigma_2: \mathcal{M}_2 \longrightarrow \mathcal{L}_2$ such that
$\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1} = \left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right) \circ \sigma_2$ (2)
As follows:

- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.

- $\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(3)} x(v) \longmapsto \frac{1}{3}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(4)} x(v) \longmapsto \frac{1}{6}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x(v) \longmapsto \frac{1}{10}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(6)} x(v) \longmapsto \frac{1}{15}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(7)} x(v) \longmapsto \frac{1}{21}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y(v) \longmapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_8 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x(v) \longmapsto \frac{1}{3}\ (\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{31}(v) - \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(2)}(v))$ ; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x(v) \longmapsto \frac{1}{12}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21} \partial_{31}(v) - \frac{1}{4} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(3)}(v)$ ; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x(v) \longmapsto \frac{1}{30}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{1}{5} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x(v) \longmapsto \frac{1}{60}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x(v) \longmapsto \frac{1}{105}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{1}{7} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x(v) \longmapsto \frac{1}{168}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{1}{8} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y(v) \longmapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y(v) \longmapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x(v) \longmapsto \frac{1}{3}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{31}^{(2)}(v) - \frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}^{(2)}(v) -$
  $\frac{1}{3}\ \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(3)} \partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x(v) \longmapsto \frac{1}{9}\ \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21} \partial_{31}^{(2)}(v) - \frac{7}{90} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}^{(2)}(v) - \frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(4)} \partial_{32}(v)$;
  where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(6)} x(v) \longmapsto \frac{1}{18} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}^{(2)}(v) - \frac{2}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32}^{(2)}(v) -$
  $\frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x(v) \longmapsto \frac{1}{30} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}^{(2)}(v) - \frac{1}{35} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32}^{(2)}(v) - \frac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)} \partial_{32}(v)$;
  where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(8)} x(v) \longmapsto \frac{1}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}^{(2)}(v) - \frac{1}{63} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32} \partial_{31}(v)$, where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(9)} x(v) \longmapsto \frac{1}{63} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{31} z(v) \longmapsto \frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{32}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$.

**Proposition (4. 2):**

The map $\sigma_2$ defined above satisfies (4.2).

**Proof:** We can see that for some terms:

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})(\mathcal{Z}_{21} x \mathcal{Z}_{21} x(v))$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$

$= \sigma_1\left(2\mathcal{Z}_{21}^{(2)}x(v)\right) - \mathcal{Z}_{21}x\partial_{21}(v) = \frac{2}{2}\mathcal{Z}_{21}x\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{21}(v) = 0.$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
$= \sigma_1\left(3\mathcal{Z}_{21}^{(3)}x(v) - \mathcal{Z}_{21}^{(2)}x\partial_{21}(v)\right) = \frac{3}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{21}(v) = 0.$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
$= \sigma_1\left(6\mathcal{Z}_{21}^{(4)}x(v) - \mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}(v)\right) = \frac{6}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v) - \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(2)}(v) = 0.$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
$= \sigma_1\left(10\mathcal{Z}_{21}^{(5)}x(v) - \mathcal{Z}_{21}^{(3)}x\partial_{21}^{(2)}(v)\right) = 2\,\mathcal{Z}_{21}x\partial_{21}^{(4)}(v) - \frac{6}{3}\mathcal{Z}_{21}x\partial_{21}^{(4)}(v) = 0.$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$
$= \sigma_1\left(\mathcal{Z}_{21}^{(3)}x\partial_{32}(v) + \mathcal{Z}_{21}^{(2)}x\partial_{31}(v)\right) - \mathcal{Z}_{32}y\partial_{21}^{(3)}(v) = \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v) + \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(3)}(v).$
And
$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)\right)$
$= \frac{1}{3}\sigma_1\left(\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v) + \mathcal{Z}_{21}^{(2)}x\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{31}\partial_{21}(v) - \mathcal{Z}_{32}y\partial_{21}^{(3)}(v)$
$= \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v) + \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(3)}(v).$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y(v)\right)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_8 \otimes \mathcal{D}_1$
$= \sigma_1\left(2\,\mathcal{Z}_{32}^{(2)}y(v)\right) - \mathcal{Z}_{32}y\partial_{32}(v) = \frac{2}{2}\mathcal{Z}_{32}y\partial_{32}(v) - \mathcal{Z}_{32}y\partial_{32}(v) = 0.$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$
$= \sigma_1\left(\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v)\right) + \mathcal{Z}_{21}x\partial_{31}^{(2)}(v) - \sigma_1\left(\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(3)}(v)\right)$
$= \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) + \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}x\partial_{31}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(3)}(v).$
And
$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)\right)$
$= \sigma_1\left(\frac{1}{3}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{31}\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \sigma_1\left(\frac{1}{3}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(2)}(v)\right) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}(v) +$
$\frac{1}{3}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(2)}(v) = \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) + \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}x\partial_{31}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(3)}(v).$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
$= \mathcal{Z}_{21}^{(8)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(7)}x\partial_{32}\partial_{31}(v) + \sigma_1\left(+\mathcal{Z}_{21}^{(6)}x\partial_{31}^{(2)}(v) - \mathcal{Z}_{32}^{(2)}y\partial_{21}^{(8)}(v)\right) = \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(8)}(v).$
And
$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{168}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{1}{8}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)\right) = \frac{1}{168}\sigma_1\left(\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v) +\right.$
$\left.2\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)\right) + \frac{1}{168}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(5)}\partial_{31}(v) -$
$\frac{1}{168}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v) - \sigma_1\left(\frac{1}{8}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(7)}(v)\right) + \frac{1}{8}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}(v) + \frac{1}{8}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(7)}(v)$
$= \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(8)}(v).$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y(v)\right)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$
$= \sigma_1\left(3\,\mathcal{Z}_{32}^{(3)}y(v) - \mathcal{Z}_{32}^{(2)}y\partial_{32}(v)\right) = \frac{3}{3}\mathcal{Z}_{32}y\partial_{32}^{(2)}(v) - \frac{2}{2}\mathcal{Z}_{32}y\partial_{32}^{(2)}(v) = 0.$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_5\otimes\mathcal{D}_0$

$=\sigma_1\left(\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v)+\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v)\right)+\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-\sigma_1\left(\mathcal{Z}_{32}^{(3)}y\partial_{21}^{(4)}(v)\right)$

$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+$
$\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v).$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)\right)$

$=\sigma_1\left(\frac{1}{3}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v)\right)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{31}\partial_{31}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-$
$\frac{1}{6}\sigma_1\left(\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}\partial_{32}^{(2)}(v)\right)-\frac{1}{6}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(2)}\partial_{32}^{(2)}(v)+$
$\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\sigma_1\left(\frac{1}{3}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)\right)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)+\frac{1}{3}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(3)}\partial_{32}(v)$

$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+$
$\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v).$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(9)}x(v)\right)$; where $v\in\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$

$=\mathcal{Z}_{21}^{(9)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(8)}x\partial_{32}^{(2)}\partial_{31}(v)+\mathcal{Z}_{21}^{(7)}x\partial_{32}\partial_{31}^{(2)}(v)+\sigma_1\left(\mathcal{Z}_{21}^{(6)}x\partial_{31}^{(3)}(v)-\mathcal{Z}_{32}^{(3)}y\partial_{21}^{(9)}(v)\right)$

$=\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(7)}\partial_{31}^{(2)}(v).$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)\right)$

$=\sigma_1\left(\frac{1}{63}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}^{(2)}(v)\right)+\frac{1}{63}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{1}{63}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}^{(2)}(v)$

$=\frac{21}{126}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(3)}(v)-\frac{21}{63}\mathcal{Z}_{32}y\partial_{21}^{(7)}\partial_{31}^{(2)}(v)=\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(7)}\partial_{31}^{(2)}(v).$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z(v)\right)$; where $v\in\mathcal{D}_{10}\otimes\mathcal{D}_8\otimes\mathcal{D}_0$

$=\sigma_1\left(\mathcal{Z}_{32}^{(3)}y\partial_{21}(v)\right)-\mathcal{Z}_{21}x\partial_{32}^{(3)}(v)-\sigma_1\left(\mathcal{Z}_{32}^{(2)}y\partial_{31}(v)\right)=\frac{1}{3}\mathcal{Z}_{32}y\partial_{32}^{(2)}\partial_{21}(v)-\mathcal{Z}_{21}x\partial_{32}^{(3)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{31}(v).$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{32}(v)\right)$

$=\sigma_1\left(\frac{1}{3}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{32}(v)\right)-\frac{1}{3}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{32}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{31}\partial_{32}(v)$

$=\frac{1}{3}\mathcal{Z}_{32}y\partial_{32}^{(2)}\partial_{21}(v)-\mathcal{Z}_{21}x\partial_{32}^{(3)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{31}(v)$

Now by employ $\sigma_2$ we can also define

$$\partial_3:\mathcal{L}_3\longrightarrow\mathcal{L}_2 \quad \text{as} \quad \partial_3=\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}$$

**Lemma (4.3):**

The composition $\partial_2\partial_3$ equal to zero.

**Proof:**

$\partial_2\partial_3(a)=\left(\delta_{\mathcal{L}_2\mathcal{L}_1}(a)+(\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1})(a)\right)\circ\left(\delta_{\mathcal{L}_3\mathcal{L}_2}(a)+(\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2})(a)\right)$

$=(\delta_{\mathcal{L}_2\mathcal{L}_1}\circ\delta_{\mathcal{L}_3\mathcal{L}_2})(a)+(\delta_{\mathcal{L}_2\mathcal{L}_1}\circ\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2})(a)+(\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\circ\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2})(a).$

But $\delta_{\mathcal{L}_2\mathcal{L}_1}\circ\sigma_2+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\circ\sigma_2=\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}$ so we get:

$\partial_2\partial_3(a)=(\delta_{\mathcal{L}_2\mathcal{L}_1}\circ\delta_{\mathcal{L}_3\mathcal{L}_2})(a)+(\delta_{\mathcal{M}_2\mathcal{L}_1}\circ\delta_{\mathcal{L}_3\mathcal{M}_2})(a)+(\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{L}_1}\circ\delta_{\mathcal{L}_3\mathcal{L}_2})(a)(\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\circ\delta_{\mathcal{L}_2\mathcal{M}_2})(a).$

By properties of the boundary map $\delta$ we get $\partial_2\partial_3=0$

We need the definition of a map $\sigma_3:\mathcal{M}_3\longrightarrow\mathcal{L}_3$ such that

$$\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}=\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\circ\sigma_3 \quad (3)$$

As follows:

- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$ ; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$ ; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(5)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x(v) \longmapsto -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x(v) \longmapsto -\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x(v) \longmapsto -\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x(v) \longmapsto -\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x(v) \longmapsto -\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x(v) \longmapsto -\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.

- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(5)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y(v) \longmapsto 0$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(4)} x(v) \longmapsto \frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21} \partial_{31}(v) - \frac{1}{4} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(2)} \partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x(v) \longmapsto \frac{1}{12} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{7}{60} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(6)} x(v) \longmapsto \frac{1}{20} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{1}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(7)} x(v) \longmapsto \frac{1}{30} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{3}{70} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(8)} x(v) \longmapsto \frac{1}{42} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{5}{168} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(9)} x(v) \longmapsto \frac{1}{56} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x(v) \longmapsto \frac{-1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(2)} \partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x(v) \longmapsto \frac{-1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x(v) \longmapsto \frac{-1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x(v) \longmapsto \frac{-1}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x(v) \longmapsto \frac{-1}{21} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(9)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21} x(v) \longmapsto -\frac{1}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{1}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v) \longmapsto -\frac{1}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{1}{45} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto -\frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v) \longmapsto -\frac{1}{30} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{1}{90} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto -\frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{4}{45} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto -\frac{2}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}(v) - \frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21} x(v) \longmapsto -\frac{1}{45} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{2}{315} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto -\frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{1}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto -\frac{5}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto -\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{5}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(8)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto -\frac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$ ; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto -\frac{1}{2} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{7}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto -\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(5)} x(v) \longmapsto -\frac{5}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{7}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$.

- $Z_{32}yZ_{31}zZ_{21}^{(2)}x(v) \longmapsto \frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$.
- $Z_{32}yZ_{31}zZ_{21}^{(3)}x(v) \longmapsto \frac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$.
- $Z_{32}yZ_{31}zZ_{21}^{(4)}x(v) \longmapsto \frac{1}{10}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$.
- $Z_{32}yZ_{31}zZ_{21}^{(5)}x(v) \longmapsto \frac{1}{15}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$.
- $Z_{32}yZ_{31}zZ_{21}^{(6)}x(v) \longmapsto \frac{1}{21}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$.
- $Z_{32}yZ_{31}zZ_{21}^{(7)}x(v) \longmapsto \frac{1}{28}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$.
- $Z_{32}yZ_{32}yZ_{31}z(v) \longmapsto 0$ ; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(2)}x(v) \longmapsto \frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{31}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(3)}x(v) \longmapsto \frac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}\partial_{31}(v) - \frac{1}{12}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(4)}x(v) \longmapsto \frac{1}{9}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{7}{90}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(5)}x(v) \longmapsto \frac{1}{12}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{15}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(6)}x(v) \longmapsto \frac{1}{15}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{2}{35}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(7)}x(v) \longmapsto \frac{1}{18}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{25}{504}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$.
- $Z_{32}^{(2)}yZ_{31}zZ_{21}^{(8)}x(v) \longmapsto \frac{1}{21}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(6)}\partial_{31}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

**Proposition (4.4):**

The map $\sigma_3$ defined above satisfies (4.3).

**Proof:** We can see that for some terms

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})(Z_{21}xZ_{21}xZ_{21}x(v))$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$

$= \sigma_2\left(2\, Z_{21}^{(2)}xZ_{21}x(v) - 2\, Z_{21}xZ_{21}^{(2)}x(v) + Z_{21}xZ_{21}x\partial_{21}(v)\right) = 0.$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(Z_{21}xZ_{21}^{(2)}xZ_{21}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_2\left(3\, Z_{21}^{(3)}xZ_{21}x(v) - 3\, Z_{21}xZ_{21}^{(3)}x(v) + Z_{21}xZ_{21}^{(2)}x\partial_{21}(v)\right) = 0.$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(Z_{32}yZ_{32}yZ_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$

$= \sigma_2\left(2\, Z_{32}^{(2)}yZ_{21}^{(4)}x(v) - Z_{32}yZ_{21}^{(4)}x\partial_{32}(v) - Z_{32}yZ_{21}^{(3)}x\partial_{31}(v) + Z_{32}yZ_{32}y\partial_{21}^{(4)}(v)\right)$
$= -\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}(v) - \frac{1}{2}Z_{32}yZ_{31}z\partial_{21}^{(3)}(v) - \frac{1}{6}\, Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v).$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}(v)\right)$
$= \sigma_2\left(\frac{1}{6}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}(v)\right) - \frac{1}{6}\, Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}(v) + \sigma_2\left(\frac{1}{6}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}(v)\right) - \frac{3}{6}\, Z_{32}yZ_{31}z\partial_{21}^{(3)}(v)$
$= -\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}(v) - \frac{1}{2}Z_{32}yZ_{31}z\partial_{21}^{(3)}(v) - \frac{1}{6}\, Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v).$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(Z_{32}^{(2)}yZ_{21}^{(3)}xZ_{21}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$

$= \sigma_2\left(Z_{21}^{(3)}xZ_{21}x\partial_{32}^{(2)}(v) + Z_{21}^{(2)}xZ_{21}x\partial_{32}\partial_{31}(v) + Z_{21}xZ_{21}x\partial_{31}^{(2)}(v) - 4\, Z_{32}^{(2)}yZ_{21}^{(4)}x(v) + Z_{32}^{(2)}yZ_{21}^{(3)}x\partial_{21}(v)\right)$
$= -\frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}(v) + Z_{32}yZ_{31}z\partial_{21}^{(3)}(v) + \frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{31}\partial_{21}(v) - \frac{3}{3}Z_{32}yZ_{31}z\partial_{21}^{(3)}(v) = 0.$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(Z_{32}^{(2)}yZ_{32}yZ_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$

$= \sigma_2\left(3\, Z_{32}^{(3)}yZ_{21}^{(4)}x(v) - Z_{32}^{(2)}yZ_{21}^{(4)}x\partial_{32}(v) - Z_{32}^{(2)}yZ_{21}^{(3)}x\partial_{31}(v) + Z_{32}^{(2)}yZ_{32}y\partial_{21}^{(4)}(v)\right)$
$= \frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{31}^{(2)}(v) - \frac{1}{2}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) - \frac{3}{4}Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{32}(v) - \frac{1}{12}\, Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v) +$
$\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(2)}\partial_{31}(v).$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v) - \frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v)\right)$
$= \sigma_2\left(-\frac{1}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}\partial_{31}(v)\right) + \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}\partial_{31}(v) + \sigma_2\left(\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}\partial_{31}(v)\right) +$
$\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}\partial_{31}(v) +$
$\sigma_2\left(\frac{1}{4}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right) - \frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{32}(v) +$
$\sigma_2\left(\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right) - \frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{32}(v)$
$= \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) - \frac{3}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v) - \frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v) +$
$\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}\partial_{31}(v)$.

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$= \sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(8)}x(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x\partial_{32}\partial_{31}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x\partial_{31}^{(2)}(v) + \mathcal{Z}_{32}y\,\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(8)}(v)\right)$
$= \frac{3}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{3}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)\ \frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) = 0.$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$= \sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(9)}x(v)\right) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(9)}x\partial_{32}^{(2)}(v) - \sigma_2\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x\partial_{32}\partial_{31}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x\partial_{31}^{(2)}(v) + \mathcal{Z}_{32}y\,\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(9)}(v)\right)$
$= \frac{3}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) - \frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(5)}\partial_{32}^{(2)}(v) = 0.$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$

$= \sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}^{(2)}(v) +\right.$
$\left.\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{31}^{(3)}(v) - 5\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x(v) + \mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}(v)\right)$
$= -\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v) - \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v) -$
$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v).$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{-1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)\right)$
$= \sigma_2\left(\frac{1}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right) - \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{31}(v) +$
$\sigma_2\left(\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right) - \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{31}(v) +$
$\sigma_2\left(\frac{1}{18}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right) - \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{32}(v) +$
$\sigma_2\left(\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right) - \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)$
$= -\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v) - \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v) -$
$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v).$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(3)}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}^{(2)}(v) +\right.$
$\left.\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(3)}(v) - 56\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(8)}x(v) + \mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\partial_{21}^{(3)}(v)\right)$
$= -\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{4}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{14}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{7}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v).$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{-5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$

$$= \sigma_2\left(\frac{5}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v) +$$
$$\sigma_2\left(\frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{31}(v) +$$
$$\sigma_2\left(\frac{2}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v) +$$
$$\sigma_2\left(\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$$
$$= -\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{4}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{14}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$$
$$-\frac{7}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v).$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$
$$= \sigma_2\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\partial_{32}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(3)}(v)\right) + \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)$$
$$= \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) + \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v).$$
And
$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}(v)\right)$$
$$= \sigma_2\left(-\frac{1}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}(v) - \sigma_2\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}(v)$$
$$= \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) + \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v).$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{31}z(v)\right)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_8 \otimes \mathcal{D}_0$
$$= \sigma_2\left(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z(v) - 2\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z(v) + \mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{31}(v)\right) = 0.$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$
$$= \sigma_2\left(-\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(3)}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)\right)$$
$$= \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{31}(v).$$
And
$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{31}(v)\right)$$
$$= \sigma_2\left(-\frac{1}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v) -$$
$$\sigma_2\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{31}(v) = \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{31}(v).$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
$$= \sigma_2\left(6\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(9)}x(v)\right) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(8)}x\partial_{32}^{(3)}(v) + \sigma_2\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x\partial_{32} - \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(9)}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)\right)$$
$$= \frac{2}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v).$$
And
$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v) +\right.$$
$$= \sigma_2\left(-\frac{1}{21}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right) + \frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{31}(v) -$$
$$\sigma_2\left(\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right) + \frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{31}(v) -$$
$$\sigma_2\left(\frac{1}{36}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right) + \frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{32}(v) -$$
$$\sigma_2\left(\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right) + \frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{32}(v)$$
$$= \frac{2}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)\,.$$

Eventually, we define the boundary maps in the complex:

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0 \; ; \qquad (4)$$

where $\partial_1$ is the operation of indicated polarization operators, $\partial_1$, $\partial_2$ and $\partial_3$ defined as follows:

- $\partial_1(\mathcal{Z}_{21}x(v)) = \partial_{21}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$.
- $\partial_1(\mathcal{Z}_{32}y(v)) = \partial_{32}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_7 \otimes \mathcal{D}_2$.
- $\partial_2(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x(v)) = \frac{1}{2}\,\mathcal{Z}_{21}x\partial_{21}\partial_{32}(v) + \mathcal{Z}_{21}x\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_5 \otimes \mathcal{D}_2$.
- $\partial_2(\mathcal{Z}_{32}y\mathcal{Z}_{31}z(v)) = \frac{1}{2}\,\mathcal{Z}_{32}y\partial_{32}\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}y\partial_{31}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_7 \otimes \mathcal{D}_1$.
- $\partial_3(\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x(v)) = \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$.

**Theorem (4.5):**

The complex (4.4.4) is exact and in characteristic-zero gives a resolution of $K_{(9,6,3)}(\mathcal{F})$.

**Proof:**

First, we prove the exactness of the complex

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1$$

Since one component of the map $\partial_3$ is a diagonalization of $\mathcal{D}_2$ into $\mathcal{D}_1 \otimes \mathcal{D}_1$ it is clear that $\partial_3$ is injective. To prove the exactness at $\mathcal{L}_2$.

For this, we need to show that:

If $v \in \ker(\partial_2)$ then $\exists\, w \in \mathcal{L}_3$ such that $\partial_3(w) = v$.

If $\partial_2(v) = 0$ then $\exists\, (a,b) \in \mathcal{L}_3 \oplus \mathcal{M}_3$ such that

$\delta(a,b) = (v,0) \in \mathcal{L}_2 \oplus \mathcal{M}_2$, but

$\delta(a,b) = \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_2\mathcal{L}_2}(b) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b)$. So we get:

$\delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{M}_3\mathcal{L}_2}(b) = v \qquad \ldots(1),$

and

$\delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b) = 0 \qquad \ldots(2)$

Now if $w = a + \sigma_3(b)$ we can see that $\partial_3(w) = v$ in fact

$\partial_3(a) = \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}(a)$, and

$\partial_3(\sigma_3(b)) = \delta_{\mathcal{M}_3\mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}(b)$, so

$$\begin{aligned}\partial_3(a + \sigma_3(b)) &= \partial_3(a) + \partial_3(\sigma_3(b)) \\ &= \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_2\mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}(b) \\ &= \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{M}_3\mathcal{L}_2}(b) + \sigma_2 \circ \Big((\delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b)\Big).\end{aligned}$$

Hence from (1) and (2), we get $\partial_3(w) = v$ ; where $w = a + \partial_3(b)$.

This proves the exactness at $\mathcal{L}_2$.

As the same way we can prove the exactness at $\mathcal{L}_1$.

Finally, from Theorem (2.3.6) we get the complex:

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0 \longrightarrow \mathcal{K}_{(9,6,3)}(\mathcal{F}) \longrightarrow 0,$$

is exact. ■

## Conflicts Of Interest

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