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Research Article

# New Generalized Extended Hardy-type Dynamic Inequalities on Time Scales

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**ABSTRACT**

This research will demonstrate several novel dynamic inequalities of the Hardy type on time scales. These inequalities will be generalized and expanded. Several new integral and difference inequalities are produced as a consequence of our findings in both continuous and discrete scenarios. The dynamic *Hölder* inequality, the integration by parts formula on time scales, and Keller's chain rule on time scales will be utilized to demonstrate the primary findings. To apply the primary findings, we shall employ discrete calculus, quantum calculus, and continuous calculus and treat them as special cases.

## 1. INTRODUCTION

The extended Hardy-type inequalities, among the most famous and widely utilized inequalities in mathematics, are the primary focus of this work. They are crucial in a wide variety of mathematical physics and mathematical analytic applications. Hardy inequalities, variants, and generalizations have a wealth of theory and a mountain of literature. They also play an important role in the investigation of inequalities that are associated with the eigenvalues of particular differential operators [1, 2]. Researchers can make predictions, analyze solutions to differential equations carefully, and see patterns via these inequalities. Theoretically and in practice, Hardy-type inequalities are crucial tools for understanding mathematical structures and for advancing research with broad scientific and engineering applications. Throughout this article, the set of real numbers is denoted by $\mathbb{R}$, while $\mathbb{R}_1 = [0, \infty)$ is the subset of $\mathbb{R}$.

### 1.1 Aims

Hardy-type inequalities aim to establish relationships and provide estimates for integrals involving functions and their derivatives. In mathematical analysis, these inequalities are essential, especially in fields such as partial differential equations and functional analysis. Hardy-type inequalities give limitations and conditions on the convergence of integrals, which help us to understand mathematical models in engineering and physics. They are useful in both applied and theoretical mathematics because they can be effective instruments for demonstrating the existence of solutions to a wide range of differential equations. All things considered, the main goal of Hardy-type inequalities is to provide a framework for the analysis and bounding of specific integral types, thereby promoting a better understanding of mathematical structures and their applications. In 1920, Hardy proved this famous distinct inequality [3].

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**Theorem 1.** *Suppose $\{b(m)\}_{m=0}^{\infty}$ is a $\mathbb{R}_1$ real sequence and $q > 1$. After that,*

$$\sum_{m=1}^{\infty}\Big(\frac{1}{m}\sum_{j=1}^{m} b(j)\Big)^q \leq \Big(\frac{q}{q-1}\Big)^q \sum_{m=1}^{\infty} b^q(m). \tag{1}$$

To streamline the existing demonstration of Hilbert's inequality for double series, Hardy stumbled onto this inequality. The integral equivalent of inequality (1) above was provided by Hardy himself in 1925 through the utilization of the calculus of variations [4].

**Theorem 2.** *Suppose $g$ is a $\mathbb{R}_1$ continuous function on $\mathbb{R}_1$ and $q > 1$. After that*

$$\int_0^{\infty}\Big(\frac{1}{y}\int_0^{y} g(s)ds\Big)^q dy \leq \Big(\frac{q}{q-1}\Big)^q \int_0^{\infty} g^q(y)dy. \tag{2}$$

In 1927, this discrete inequality was derived from an extension of the inequality (1) by Littlewood and Hardy [5].

**Theorem 3.** *Suppose that series of $\mathbb{R}_1$ real numbers $\{b(m)\}_{m=0}^{\infty}$ exists.*
*$(i)$ If $\beta > 1$ and $q > 1$. After that,*

$$\sum_{m=1}^{\infty}\frac{1}{m^{\beta}}\Big(\sum_{j=1}^{m} b(j)\Big)^q \leq C(q,\beta)\sum_{m=1}^{\infty}\frac{1}{m^{\beta-q}} b^q(m). \tag{3}$$

*$(ii)$ If $\beta < 1$ and $q > 1$. After that,*

$$\sum_{m=1}^{\infty}\frac{1}{m^{\beta}}\Big(\sum_{j=m}^{\infty} b(j)\Big)^q \leq C(q,\beta)\sum_{m=1}^{\infty}\frac{1}{m^{\beta-q}} b^q(m), \tag{4}$$

where the nonnegative constant $C(q,\beta)$ is defined in inequality (3) and (4) and is dependent on $q$ and $\beta$. This notable disparity in Hardy-type was shown by the authors in the same work [5].

**Theorem 4.** *Suppose $g$ is a $\mathbb{R}_1$ continuous function on $\mathbb{R}_1$ and $q > 1$. After that*

$$\int_0^{\infty}\Big(\frac{1}{y}\int_y^{\infty} g(s)ds\Big)^q dy \leq q^q \int_0^{\infty} g^q(y)dy,$$

*which can be rewritten as*

$$\int_0^{\infty}\Big(\int_y^{\infty} g(s)ds\Big)^q dy \leq q^q \int_0^{\infty} y^q g^q(y)dy. \tag{5}$$

Hardy [6] showed that the following formulations of the inequalities (3) and (4) are continuous in the year 1928.

**Theorem 5.** *Suppose $g$ is a $\mathbb{R}_1$ continuous function on $\mathbb{R}_1$.*
*$(i)$ If $\beta > 1$ and $q > 1$. After that,*

$$\int_0^{\infty}\frac{1}{y^{\beta}}\Big(\int_0^{y} g(s)ds\Big)^q dy \leq \Big(\frac{q}{\beta-1}\Big)^q \int_0^{\infty}\frac{1}{y^{\beta-q}} g^q(y)dy. \tag{6}$$

*$(ii)$ If $\beta < 1$ and $q > 1$. After that,*

$$\int_0^{\infty}\frac{1}{y^{\beta}}\Big(\int_y^{\infty} g(s)ds\Big)^q dy \leq \Big(\frac{q}{1-\beta}\Big)^q \int_0^{\infty}\frac{1}{y^{\beta-q}} g^q(y)dy. \tag{7}$$

The discrete Hardy inequality in (1) was expanded by Copson, who also produced the two subsequent discrete inequalities in the same year [7].

**Theorem 6.** *Suppose* $\{b(m)\}_{m=1}^{\infty}$ *and* $\{h(m)\}_{m=1}^{\infty}$ *be sequences of* $\mathbb{R}_1$. *Then*

$$\sum_{m=1}^{\infty} \frac{h(m)(\sum_{j=1}^{m} h(j)b(j))^q}{(\sum_{j=1}^{m} h(j))^{\beta}} \leq \Big(\frac{q}{\beta-1}\Big)^q \sum_{m=1}^{\infty} h(m)b^q(m)\Big(\sum_{j=1}^{m} h(j)\Big)^{q-\beta}, \tag{8}$$

*for* $q \geq \beta > 1$, *and*

$$\sum_{m=1}^{\infty} \frac{h(m)(\sum_{j=m}^{\infty} h(j)b(j))^q}{(\sum_{j=1}^{m} h(j))^{\beta}} \leq \Big(\frac{q}{1-\beta}\Big)^q \sum_{m=1}^{\infty} h(m)b^q(m)\Big(\sum_{j=1}^{m} h(j)\Big)^{q-\beta}, \tag{9}$$

*for* $q > 1 > \beta \geq 0$.

In year 1970, Leindler [8] examined the scenario in which the sum $\sum_{m=1}^{\infty} h(j) < \infty$ on the side to the left of the inequality (8) is replaced by the sum $\sum_{n=m}^{\infty} h(j) < \infty$. His result can be expressed in the next theorem.

**Theorem 7.** *Let* $\{b(m)\}_{m=1}^{\infty}$ *and* $\{h(m)\}_{m=1}^{\infty}$ *be sequences of* $\mathbb{R}_1$ *with* $\sum_{j=m}^{\infty} h(j) < \infty$. *Suppose* $q > 1 > \beta \geq 0$, *then*

$$\sum_{m=1}^{\infty} \frac{h(m)(\sum_{j=1}^{m} h(j)b(j))^q}{(\sum_{j=m}^{\infty} h(j))^{\beta}} \leq \Big(\frac{q}{1-\beta}\Big)^q \sum_{m=1}^{\infty} h(m)b^q(m)\Big(\sum_{j=m}^{\infty} h(j)\Big)^{q-\beta}, \tag{10}$$

Copson [9] provided the continuous versions of the inequality (8) and (9) in 1976. Specifically, he demonstrated the following result through his work.

**Theorem 8.** *Suppose* $g$ *and* $h$ *are* $\mathbb{R}_1$ *continuous functions on* $\mathbb{R}_1$. *Then*

$$\int_0^{\infty} \frac{h(y)(\int_0^y h(s)g(s)ds)^q}{(\int_0^y h(s)ds)^{\beta}} dy \leq \Big(\frac{q}{\beta-1}\Big)^q \int_0^{\infty} h(y)g^q(y)\Big(\int_0^y h(s)ds\Big)^{q-\beta} dy, \tag{11}$$

*for* $q \geq \beta > 1$,
*and*

$$\int_0^{\infty} \frac{h(y)(\int_y^{\infty} h(s)g(s)ds)^q}{(\int_0^y h(s)ds)^{\beta}} dy \leq \Big(\frac{q}{1-\beta}\Big)^q \int_0^{\infty} h(y)g^q(y)\Big(\int_0^y h(s)ds\Big)^{q-\beta} dy, \tag{12}$$

*for* $q > 1 > \beta \geq 0$.

Bennet showed the following result in 1987 [10], utilizing the work that Leindler had done in Theorem 1.7

**Theorem 9.** *Suppose* $\{b(m)\}_{m=1}^{\infty}$ *and* $\{h(m)\}_{m=1}^{\infty}$ *be sequences of* $\mathbb{R}_1$ *with* $\sum_{j=m}^{\infty} h(j) < \infty$. *Provided that* $q \geq \beta > 1$, *then:*

$$\sum_{m=1}^{\infty} \frac{h(m)(\sum_{j=m}^{\infty} h(j)b(j))^q}{(\sum_{j=m}^{\infty} h(j))^{\beta}} \leq \Big(\frac{q}{\beta-1}\Big)^q \sum_{m=1}^{\infty} h(m)b^q(m)\Big(\sum_{j=m}^{\infty} h(j)\Big)^{q-\beta}, \tag{13}$$

Many researchers have been and continue to be interested in studying Hardy-type inequalities. We direct the interested reader to the articles [6–15], the books [16–19], and the references provided therein for a wealth of information on the aforementioned inequalities, which have been extensively studied and improved for several decades.

In an effort to bridge the gap between discrete and continuous analysis, Stefan Hilger launched the widely discussed theory of time scales in his doctoral dissertation. Problems in a variety of domains, including physics, engineering, and economics, can be analysed and generalized thanks to this capability of researchers [20, 21].

The primary objective is to solve a dynamic inequality or equation in situations where the boundaries of the unresolved function are the so-called time scale $\mathbb{T}$. This time scale can be any closed subset of the real numbers $\mathbb{R}$ [21–23].

Calculus on time scales is most famously used in three areas: differential calculus, difference calculus, and quantum calculus [23], *i.e.*, at the same time $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{n^{\mathbb{Z}}} = \{n^{Z} : z \in \mathbb{Z}\} \cup \{0\}$ where $n > 1$. A large portion of time scale calculus is organized and summarized in the book on time scales by Bohner and Peterson [24–26].

Before anyone else, *Řehák* extended Hardy-type inequalities with time scales in 2005. As will be seen below, he consolidated his previous work on the oscillation theory of half-linear dynamic equations which involved generalizing inequalities (1) and (2) to any arbitrary time scale into a single form [27].

**Theorem 10.** *Suppose $\mathbb{T}$ is a time scale, $g \in C_{rd}([b,\infty)_{\mathbb{T}}, \mathbb{R}_1)$. In this theorem $q > 1$,*

$$\int_b^\infty \Big(\frac{\int_b^{\xi(\zeta)} g(s)\Delta s}{\xi(\zeta) - b}\Big)^q \Delta\zeta < \Big(\frac{q}{q-1}\Big)^q \int_b^\infty g^q(\zeta)\Delta\zeta, \tag{14}$$

*unless $g \equiv 0$.*

Additionally, the inequality (14) becomes sharp when $\mu(\zeta)/\zeta \to 0$ as $\zeta \to \infty$. Saker et al. [28] demonstrated four Hardy-type dynamic inequalities in 2014.

**Theorem 11.** *Consider $\mathbb{T}$ is a time scale with $b \in [0,\infty)_{\mathbb{T}}$. suppose $g$ and $h$ are $\mathbb{R}_1$ rd-continuous functions at $[b,\infty)_{\mathbb{T}}$.*
*(i) If $q \geq \beta > 1$, then*

$$\int_b^\infty \frac{h(\zeta)\big(\int_b^{\xi(\zeta)} h(s)g(s)\Delta s\big)^q}{\big(\int_b^{\xi(\zeta)} h(s)\Delta s\big)^\beta}\Delta\zeta \leq \Big(\frac{q}{\beta-1}\Big)^q \int_b^\infty \frac{h(\zeta)g^q(\zeta)\big(\int_b^{\xi(\zeta)} h(s)\Delta s\big)^{\beta(q-1)}}{\big(\int_b^{\zeta} h(s)\Delta s\big)^{q(\beta-1)}}\Delta\zeta. \tag{15}$$

*(ii) If $q > 1 > \beta \geq 0$, then*

$$\int_b^\infty \frac{h(\zeta)\big(\int_\zeta^{\infty} h(s)g(s)\Delta s\big)^q}{\big(\int_b^{\xi(\zeta)} h(s)\Delta s\big)^\beta}\Delta\zeta \leq \Big(\frac{q}{1-\beta}\Big)^q \int_b^\infty h(\zeta)g^q(\zeta)\Big(\int_b^{\xi(\zeta)} h(s)\Delta s\Big)^{q-\beta}\Delta\zeta. \tag{16}$$

*(iii) If $q > 1 > \beta \geq 0$, then*

$$\int_b^\infty \frac{h(\zeta)\big(\int_b^{\xi(\zeta)} h(s)g(s)\Delta s\big)^q}{\big(\int_\zeta^{\infty} h(s)\Delta s\big)^\beta}\Delta\zeta \leq \Big(\frac{q}{1-\beta}\Big)^q \int_b^\infty h(\zeta)g^q(\zeta)\Big(\int_\zeta^{\infty} h(s)\Delta s\Big)^{q-\beta}\Delta\zeta. \tag{17}$$

*(iv) If $q \geq \beta > 1$, then*

$$\int_b^\infty \frac{h(\zeta)\big(\int_\zeta^{\infty} h(s)g(s)\Delta s\big)^q}{\big(\int_\zeta^{\infty} h(s)\Delta s\big)^\beta}\Delta\zeta \leq \Big(\frac{q}{\beta-1}\Big)^q \int_b^\infty h(\zeta)g^q(\zeta)\Big(\int_\zeta^{\infty} h(s)\Delta s\Big)^{q-\beta}\Delta\zeta. \tag{18}$$

Ahmed A. El-Deeb proved some novel Hardy-type dynamic inequalities over arbitrary time scales in 2020 and 2022, respectively [2, 29]. The results stated above apply to both continuous and discrete domains. In light of the existing inequalities based on time scales, the present study seeks to offer new, more general results. Thus, supreme outcomes would be generated, from which several other results, both past and present, can be derived. For several types of time-scale inequalities, including integrals of the Hardy-type dynamic inequalities, see the following papers [2, 29–33].

Table 1 provides a quick summary of various time scales and the Hardy-type inequalities that are connected with them. By doing so, the chart highlights the definitions, types of inequalities, and typical applications for each time scale. However, it is important to keep in mind that the particulars of these inequalities can change depending on the precise formulation and properties of the dynamic equations that are being investigated on each time scale.

For a more versatile and all-encompassing study of dynamic systems, time scales offer a unified framework that integrates discrete and continuous mathematical modeling. To comprehend the actions of solutions to dynamic equations, adaptations of Hardy-type inequalities to various time scales are essential. To guarantee seamless transitions in dynamic processes,

TABLE I. Comparison of time scales with Hardy-type inequalities

| **Time Scale** | **Definition** | **Hardy-Type Inequality** | **Applications** |
|---|---|---|---|
| Continuous Time (CT) | $\mathbb{R}$ | Classical Hardy inequalities involve functions and derivatives | Analysis of differential equations and integral inequalities |
| Discrete Time (DT) | $\mathbb{Z}$ | Discrete-time Hardy inequalities for difference equations | Stability and behavior analysis in discrete-time systems |
| Time Scales (TS) | Unified framework | Generalized Hardy-type inequalities for both continuous and discrete cases | Analysis of systems with mixed continuous and discrete dynamic |
| Delta and Nabla TS | Jump operators (forward and backward) | Extension of classical Hardy inequalities to dynamic equations on time scales | Systems with forward and backward jumps |
| Hilger Time Scale | Specific type defined by Stefan Hilger | Development of Hardy-type inequalities specific to Hilger time scales | Problems where a Hilger time scale is the natural choice |
| Mixed Time Scales | Combination of different time scales | Corresponding Hardy-type inequalities developed for mixed time scales | Systems with components evolving on different time scales |

stability assessments in differential equations rely on Hardy inequalities in continuous time.

When it comes to digital signal processing and difference equations, discrete-time Hardy inequalities are useful for describing how solutions evolve. Time scales provide a holistic view by incorporating continuous and discrete processes in a seamless manner. The classical results can be extended to dynamic systems with both forward and backward jumps by using the delta and nabla time scales. For both continuous and discrete analysis, a unified approach is given by Hilger time scales with particular Hardy-type inequality. Combining different time structures on mixed time scales poses problems that are addressed by related Hardy-type inequalities, providing a flexible toolbox for analyzing systems with different temporal properties. We have high hopes that the reader has sufficient knowledge of the dynamic inequalities of the Hardy type on time scales.

In this paper, we demonstrate the existence of several hardy-type dynamic inequalities on time scales, which are generalizations of inequalities from previous papers. Some known integral inequalities of the Hardy type will be extended by the results, and some continuous inequalities and their discrete analogues will be unified and extended as well.

Here is the layout of the paper: The second section provides an overview of the fundamental ideas and lemmas of time scale calculus. In Section 3, we present the most important findings and explain their significance. Section 4 provides a discussion and conclusion of the manuscript.

## 2. BASICS OF TIME SCALES

A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$ that is arbitrary and nonempty. The topology of $\mathbb{T}$ is assumed to be the same throughout, deriving from the standard topology of the real numbers $\mathbb{R}$. The operator for forward jump $\xi : \mathbb{T} \to \mathbb{T}$ is defined as follows:

$$\xi(\zeta) := inf\{s \in \mathbb{T} : s > \zeta\}, \qquad \zeta \in \mathbb{T}. \tag{19}$$

We established in the previous definition that $inf\phi = sup\mathbb{T}$ (i.e., if $\zeta$ is the greatest of $\mathbb{T}$, then $\xi(\zeta) = \zeta$), where $\phi$ is the empty set.

On a time scale $\mathbb{T}$, consider the real-valued function $g : \mathbb{T} \to \mathbb{R}$. After that, for each $\zeta \in \mathbb{T}^k$, we define $g^\Delta(\zeta)$ as the number that, if it exists, has the condition that, $\epsilon > 0$ there is a neighborhood $\mu$ of $\zeta$ such that, $\forall\ s \in \mu$, we have

$$|[g(\xi(\zeta)) - g(s)] - g^\Delta(\zeta)[\xi(\zeta) - s]| \le \epsilon|\xi(\zeta) - s|. \tag{20}$$

In this particular scenario, we say that $g$ is delta differentiable on $\mathbb{T}^k$ lay out $g^\Delta(\zeta)$ exists, $\forall\ \zeta \in \mathbb{T}^k$. It is possible to derive the product rule for $gh$ from two delta differentiable functions $g$ and $h$ in the following way.

$$(gh)^\Delta(\zeta) = g^\Delta(\zeta)h(\zeta) + g(\xi(\zeta))h^\Delta(\zeta) = g(\zeta)h^\Delta(\zeta) + g^\Delta(\zeta)h(\xi(\zeta)). \tag{21}$$

The formula for delta integration by parts can be represented through the use of time scales as

$$\int_b^c w(\zeta)x^\Delta(\zeta)\Delta\zeta = [w(\zeta)x(\zeta)]_b^c - \int_b^c w^\Delta(\zeta)x^\xi(\zeta)\Delta\zeta. \tag{22}$$

We shall make frequent use of the following significant relations.
$(i)$ If $\mathbb{T} = \mathbb{R}$, then

$$\xi(\zeta) = \zeta, \qquad g^\Delta(\zeta) = g'(\zeta), \qquad \int_b^c g(\zeta)\Delta\zeta = \int_b^c g(\zeta)d\zeta. \tag{23}$$

$(ii)$ If $\mathbb{T} = \mathbb{Z}$, then

$$\xi(\zeta) = \zeta + 1, \qquad g^\Delta(\zeta) = \Delta g(\zeta), \qquad \int_b^c g(\zeta)\Delta\zeta = \sum_{\zeta=b}^{c-1} g(\zeta). \tag{24}$$

$(iii)$ If $\mathbb{T} = f\mathbb{Z}$, then

$$\xi(\zeta) = \zeta + f, \qquad g^\Delta(\zeta) = \frac{g(\zeta + f) - g(\zeta)}{h}, \qquad \int_b^c g(\zeta)\Delta\zeta = \sum_{\zeta=\frac{b}{f}}^{\frac{c}{f}-1} fg(f\zeta). \tag{25}$$

$(iv)$ If $\mathbb{T} = \overline{n^{\mathbb{Z}}}$, then

$$\xi(\zeta) = n\zeta, \qquad g^\Delta(\zeta) = \frac{g(n\zeta) - g(\zeta)}{(n-1)\zeta}, \qquad \int_b^c g(\zeta)\Delta\zeta = (n-1)\sum_{\zeta=log_n\zeta}^{(log_n c)-1} n^\zeta g(n^\zeta). \tag{26}$$

**Lemma 1.** *Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous function $h : \mathbb{T} \to \mathbb{R}$ is $\Delta$ differentiable on $\mathbb{T}^k$, and $g : \mathbb{R} \to \mathbb{R}$ is continuous differentiable (Chain Rule on time scales, see* [24]*). Then, in such case, $d \in [\zeta, \xi(\zeta)]$ with*

$$(goh)^\Delta(\zeta) = g'(h(d))h^\Delta(\zeta). \tag{27}$$

**Lemma 2.** *Consider $b, c \in \mathbb{T}$ and $g, h \in C_{rd}([b, c]_{\mathbb{T}}, \mathbb{R}_1)$ (Dynamic Hölder inequality, see* [28]*). If $q, n > 1$ with $1/q + 1/n = 1$, then*

$$\int_b^c g(\zeta)h(\zeta)\Delta\zeta \le \Big(\int_b^c g^q(\zeta)\Delta\zeta\Big)^{\frac{1}{q}}\big(h^n(\zeta)\Delta\zeta\big)^{\frac{1}{n}}. \tag{28}$$

## 3. MAIN RESULTS

**Theorem 12.** *Consider $\mathbb{T}$ is a time scale with $b \in [0, \infty)_{\mathbb{T}}$. consider $g, h, l, r, u, v, k, p$ be $\mathbb{R}_1$ rd-continuous functions on $[b, \infty)_{\mathbb{T}}$ that is to say $l$ is non increasing. Furthermore, let us assume that there is $\phi, \theta, \alpha, \gamma \ge 0$ such that $\frac{u^\Delta(\zeta)}{u(\zeta)} \le \phi(\frac{H^\Delta(\zeta)}{H^\xi(\zeta)})$, $\frac{p^\Delta(\zeta)}{p(\zeta)} \le \theta(\frac{H^\Delta(\zeta)}{H^\xi(\zeta)})$, $\frac{v^\Delta(\zeta)}{v(\zeta)} \le \alpha(\frac{L^\Delta(\zeta)}{L^\xi(\zeta)})$ and $\frac{k^\Delta(\zeta)}{k^\xi(\zeta)} \le \gamma(\frac{L^\Delta(\zeta)}{L(\zeta)})$, where $H(\zeta) = \int_b^\zeta h(s)\Delta s$ with $H(\infty) = \infty$ and $L(\zeta) = \int_b^\zeta r(s)g(s)\Delta s$,*

$\zeta \in [b, \infty)_{\mathbb{T}}$.
*If $q \geq 1$ and $\beta > \phi + 1$, then*

$$\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}(L^\xi(\zeta))^q\Delta\zeta$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q\int_b^\infty \frac{l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)(H^\xi(\zeta))^{\beta(q-1)}}{h^{q-1}(\zeta)H^{q(\beta-1)}(\zeta)}\Delta\zeta. \tag{29}$$

*Proof.* Applying the formula of integration by parts to time scales in (20) with

$$w^\Delta(\zeta) = p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta} \qquad and \qquad x^\xi(\zeta) = l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)(L^\xi(\zeta))^q,$$

we have

$$\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}(L^\xi(\zeta))^q\Delta\zeta$$

$$= [w(\zeta)l(\zeta)v(\zeta)k(\zeta)L^q(\zeta)]_b^\infty + \int_b^\infty (-w(\zeta))(l(\zeta)v(\zeta)k(\zeta)L^q(\zeta))^\Delta\Delta\zeta, \tag{30}$$

where we suppose that

$$w(\zeta) = -\int_\zeta^\infty p(s)u(s)h(s)(H^\xi(s))^{-\beta}\Delta s.$$

Applying the rules in (21) and (27) on time scales, and the hypothesis $\frac{u^\Delta(\zeta)}{u(\zeta)} \leq \phi(\frac{H^\Delta(\zeta)}{H^\xi(\zeta)})$, $\frac{p^\Delta(\zeta)}{p(\zeta)} \leq \theta(\frac{H^\Delta(\zeta)}{H^\xi(\zeta)})$ we see that there is $d \in [s, \xi(s)]$ such that

$$(p(s)u(s)H^{1-\beta}(s))^\Delta = p^\Delta(s)u(s)H^{1-\beta}(s) + p(s)u^\Delta(s)(H^\xi(s))^{1-\beta} + p(s)u(s)(H^{1-\beta}(s))^\Delta$$

$$\leq \theta p(s)u(s)H^\Delta(s)(H^\xi(s))^{-\beta} + \phi p(s)u(s)H^\Delta(s)(H^\xi(s))^{-\beta}$$
$$+(1-\beta)p(s)u(s)H^{-\beta}(d)H^\Delta(s).$$

Since $H^\Delta(s) = h(s) \geq 0$, $d \leq \xi(s)$ and $\beta > 1$, we get

$$(p(s)u(s)H^{1-\beta}(s))^\Delta \leq \theta p(s)u(s)h(s)(H^\xi(s))^{-\beta} + \phi p(s)u(s)h(s)(H^\xi(s))^{-\beta}$$
$$+(1-\beta)p(s)u(s)h(s)(H^\xi(s))^{-\beta}.$$

$$= (1-\beta+\theta+\phi)p(s)u(s)h(s)(H^\xi(s))^{-\beta}.$$

This gives us that

$$p(s)u(s)h(s)(H^\xi(s))^{-\beta} \leq \frac{1}{1-\beta+\theta+\phi}(p(s)u(s)H^{1-\beta}(s))^\Delta.$$

Hence

$$-w(\zeta) = \int_\zeta^\infty p(s)u(s)h(s)(H^\xi(s))^{-\beta}\Delta s \leq \frac{1}{1-\beta+\theta+\phi}\int_\zeta^\infty (p(s)u(s)(H^{1-\beta}(s))^\Delta\Delta s$$

$$= \frac{1}{\beta-\theta-\phi-1}p(\zeta)u(\zeta)(H^{1-\beta}(\zeta). \tag{31}$$

Applying the rules (21) and (27) on time scales, we may observe that there $d \in [\zeta, \xi(\zeta)]$ such that

$$\big(l(\zeta)k(\zeta)v(\zeta)L^q(\zeta)\big)^\Delta = (l(\zeta)k(\zeta)v(\zeta))^\Delta L^q(\zeta) + l^\xi(\zeta)k^\xi(\zeta)v^\xi(\zeta)(L^q(\zeta))^\Delta$$

$$= l^\Delta(\zeta)k(\zeta)v(\zeta)L^q(\zeta) + l^\xi(\zeta)k^\Delta(\zeta)v(\zeta)L^q(\zeta) + l^\xi(\zeta)k(\zeta)v^\Delta(\zeta)L^q(\zeta)$$
$$+ql^\xi(\zeta)k^\xi(\zeta)v^\xi(\zeta)L^{q-1}(d)L^\Delta(\zeta).$$

Since $l^\Delta(\zeta) \leq 0$, $L^\Delta(\zeta) = r(\zeta)g(\zeta) \geq 0$, $d \leq \xi(\zeta)$, $q \geq 1, \frac{v^\Delta(\zeta)}{v^\xi(\zeta)} \leq \alpha(\frac{L^\Delta(\zeta)}{L(\zeta)})$, and $\frac{k^\Delta(\zeta)}{k^\xi(\zeta)} \leq \gamma(\frac{L^\Delta(\zeta)}{L(\zeta)})$, we have

$$\big(l(\zeta)k(\zeta)v(\zeta)L^q(\zeta)\big)^\Delta \leq \gamma k^\xi(\zeta)l^\xi(\zeta)v^\xi(\zeta)r(\zeta)g(\zeta)L^{q-1}(\zeta) + \alpha v^\xi(\zeta)l^\xi(\zeta)k(\zeta)r(\zeta)g(\zeta)L^{q-1}(\zeta)$$
$$ql^\xi(\zeta)k^\xi(\zeta)v^\xi(\zeta)r(\zeta)g(\zeta)(L^\xi(\zeta))^{q-1}$$

$$\leq (q+\alpha+\gamma)k^\xi(\zeta)l^\xi(\zeta)v^\xi(\zeta)r(\zeta)g(\zeta)(L^\xi(\zeta))^{q-1}. \tag{32}$$

It is important to note that $L(b) = 0$ and $w(\infty) = 0$, which we receive. After combining (30), (31), and (32)

$$\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}(L^\xi(\zeta))^q\Delta\zeta$$
$$\leq \frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)r(\zeta)g(\zeta)H^{1-\beta}(\zeta)(L^\xi(\zeta))^{q-1}\Delta\zeta,$$

or equivalently,

$$\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}(L^\xi(\zeta))^q\Delta\zeta$$
$$\leq \frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\int_b^\infty (l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta))^{\frac{q-1}{q}}(H^\xi(\zeta))^{\frac{-\beta(q-1)}{q}}(L^\xi(\zeta))^{q-1})$$
$$\times\Big(\frac{(l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta))^{\frac{1}{q}}r(\zeta)g(\zeta)(H^\xi(\zeta))^{\frac{\beta(q-1)}{q}}}{h^{\frac{q-1}{q}}(\zeta)H^{\beta-1}(\zeta)}\Big)\Delta\zeta.$$

The dynamic *Hölder* inequality (28), when applied with indices $q$ and $\frac{q}{q-1}$, allows us to obtain.

$$\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}(L^\xi(\zeta))^q\Delta\zeta$$
$$\leq \frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big(\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}(L^\xi(\zeta))^q\Delta\zeta\Big)^{\frac{(q-1)}{q}}$$
$$\times\Big(\int_b^\infty \frac{l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)(H^\xi(\zeta))^{\beta(q-1)}}{h^{q-1}(\zeta)H^{q(\beta-1)}(\zeta)}\Delta\zeta\Big)^{\frac{1}{q}},$$

which implies that

$$\int_b^\infty l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}(L^\xi(\zeta))^q\Delta\zeta$$
$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q\int_b^\infty \frac{l^\xi(\zeta)v^\xi(\zeta)k^\xi(\zeta)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)(H^\xi(\zeta))^{\beta(q-1)}}{h^{q-1}(\zeta)H^{q(\beta-1)}(\zeta)}\Delta\zeta.$$

The proof is now complete.

**Remark 1.** *Assuming $\phi = \alpha = \theta = \gamma = 0$, we may simplify inequality (29) to inequality (15) by setting $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$ and $r(\zeta) = h(\zeta)$.*

**Corollary 1.** *According to Theorem 12, if $\mathbb{T} = \mathbb{R}$, then inequality (29) can be shown using relations (23)*

$$\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)h(\zeta)H^{-\beta}(\zeta)L^q(\zeta)\Delta\zeta$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \int_b^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)H^{q-\beta}(\zeta)}{h^{q-1}(\zeta)}d\zeta, \tag{33}$$

*where : $H(\zeta) = \int_b^\zeta h(s)ds$ and $L(\zeta) = \int_b^\zeta r(s)g(s)ds$.*

**Remark 2.** *Assuming $\phi = \alpha = \theta = \gamma = 0$, we may simplify inequality (33) to inequality (11) in corollary 1 by setting $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$, $r(\zeta) = h(\zeta)$ and $b = 0$.*

**Corollary 2.** *In theorem 12, we take $\mathbb{T} = f\mathbb{Z}$, and then inequality (27) becomes, using the relations (25)*

$$\sum_{\zeta=\frac{b}{f}}^{\infty} l(f\zeta+f)v(f\zeta+f)k(f\zeta+f)p(f\zeta)u(f\zeta)h(f\zeta)H^{-\beta}(f\zeta+f)L^q(f\zeta+f)$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \sum_{\zeta=\frac{b}{f}}^{\infty} \frac{l(f\zeta+f)v(f\zeta+f)k(f\zeta+f)p(f\zeta)u(f\zeta)r^q(f\zeta)g^q(f\zeta)H^{\beta(q-1)}(f\zeta+f)}{h^{q-1}(f\zeta)H^{q(\beta-1)}(f\zeta+f)}, \tag{34}$$

*where*
*$H(\zeta) = f\sum_{s=\frac{b}{f}}^{\frac{\zeta}{f}-1} h(fs)$ and $L(\zeta) = f\sum_{s=\frac{b}{f}}^{\frac{\zeta}{f}-1} r(fs)g(fs)$.*

**Corollary 3.** *In Corollary 2, we just set $f = 1$ for $\mathbb{T} = \mathbb{Z}$. Then inequality (27) becomes:*

$$\sum_{\zeta=b}^{\infty} l(\zeta+1)v(\zeta+1)k(\zeta+1)p(\zeta)u(\zeta)h(\zeta)H^{-\beta}(\zeta+1)L^q(\zeta+1)$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \sum_{\zeta=b}^{\infty} \frac{l(\zeta+1)v(\zeta+1)k(\zeta+1)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)H^{\beta(q-1)}(\zeta+1)}{h^{q-1}(\zeta)H^{q(\beta-1)}(\zeta+1)}, \tag{35}$$

*where*
*$H(\zeta) = \sum_{s=b}^{\zeta-1} h(s)$ and $L(\zeta) = \sum_{s=b}^{\zeta-1} r(s)g(s)$.*

**Remark 3.** *Assuming $\phi = \alpha = \theta = \gamma = 0$, in corollary 3, set $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$, and $r(\zeta) = h(\zeta)$ and $b = 1$, then inequality (33) reduces to:*

$$\sum_{\zeta=1}^{\infty} \frac{h(\zeta)(\sum_{s=1}^{\zeta} h(s)g(s))^q}{(\sum_{s=1}^{\zeta} h(s))^{\beta}} \leq \Big(\frac{q}{\beta-1}\Big)^q \sum_{\zeta=1}^{\infty} \frac{h(\zeta)g^q(\zeta)(\sum_{s=1}^{\zeta})h(s))^{\beta(q-1)}}{(\sum_{s=1}^{\zeta-1} h(s))^{q(\beta-1)}}, \tag{36}$$

*according to this interpretation, the discrete inequality (8) can be understood differently.*

**Corollary 4.** *Using relations (26) with Theorem 12 and using $\mathbb{T} = \overline{n^{\mathbb{Z}}}$, then inequality (29) becomes:*

$$\sum_{\zeta=log_n b}^{\infty} l(n^{\zeta+1})v(n^{\zeta+1})k(n^{\zeta+1})p(n^\zeta)u(n^\zeta)h(n^\zeta)H^{-\beta}(n^{\zeta+1})L^q(n^{\zeta+1})$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \sum_{\zeta=log_n b}^{\infty} \frac{l(n^{\zeta+1})v(n^{\zeta+1})k(n^{\zeta+1})p(n^\zeta)u(n^\zeta)r^q(n^\zeta)g^q(n^\zeta)H^{\beta(q-1)}(n^{\zeta+1})}{h^{q-1}(n^\zeta)H^{q(\beta-1)}(n^\zeta)}, \tag{37}$$

*where*
*$H(\zeta) = (n-1)\sum_{s=log_n b}^{(log_n\zeta)-1} n^s h(n^s)$ and $L(\zeta) = (n-1)\sum_{s=log_n b}^{(log_n\zeta)-1} n^s r(n^s)g(n^s)$.*

**Theorem 13.** *Consider $\mathbb{T}$ is a time scale with $b \in [0,\infty)_{\mathbb{T}}$. Inclusion, consider $g, h, l, r, u, p, v, k$ are $\mathbb{R}_1$ rd-continuous functions on $[b,\infty)_{\mathbb{T}}$ that is to say $l$ is nondecreasing. Furthermore, let us assume that there is $\phi, \theta, \alpha, \gamma \geq 0$ such that $\frac{u^{\Delta}(\zeta)}{u^{\xi}(\zeta)} \leq \phi(\frac{H^{\Delta}(\zeta)}{H(\zeta)})$, $\frac{p^{\Delta}(\zeta)}{p^{\xi}(\zeta)} \leq \theta(\frac{H^{\Delta}(\zeta)}{H(\zeta)})$, $\frac{v^{\Delta}(\zeta)}{v(\zeta)} \leq \alpha(\frac{G^{\Delta}(\zeta)}{G^{\xi}(\zeta)})$ and $\frac{k^{\Delta}(\zeta)}{k^{\xi}(\zeta)} \leq \gamma(\frac{G^{\Delta}(\zeta)}{G(\zeta)})$, where*
*$H(\zeta) = \int_b^{\zeta} h(s)\Delta s$ with $H(\infty) = \infty$ and $G(\zeta) = \int_{\zeta}^{\infty} r(s)g(s)\Delta s, \quad \zeta \in [b,\infty)_{\mathbb{T}}$.*
*If $q \geq 1$ and $0 \leq \beta < 1$, then*

$$\int_b^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)(H^{\xi}(\zeta))^{-\beta}(G^{q}(\zeta))\Delta\zeta$$

$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^{q} \int_b^{\infty} \frac{l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)r^{q}(\zeta)g^{q}(\zeta)(H^{\xi}(\zeta))^{(q-\beta)}}{h^{q-1}(\zeta)}\Delta\zeta. \tag{38}$$

*Proof.* Applying the formula of integration by parts to time scales (22) with:

$$\int_b^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)(H^{\xi}(\zeta))^{-\beta}G^{q}(\zeta)\Delta\zeta$$

$$= [w(\zeta)l(\zeta)v(\zeta)k(\zeta)G^{q}(\zeta)]_b^{\infty} + \int_b^{\infty} w^{\xi}(\zeta)(-l(\zeta)v(\zeta)k(\zeta)G^{q}(\zeta))^{\Delta}\Delta\zeta, \tag{39}$$

where we suppose that:

$$w(\zeta) = \int_b^{\zeta} p^{\xi}(s)u^{\xi}(s)h(s)(H^{\xi}(s))^{-\beta}\Delta s.$$

Applying the rules (19) and (25) on time scales, and the hypothesis $\frac{u^{\Delta}(\zeta)}{u(\zeta)} \leq \phi(\frac{H^{\Delta}(\zeta)}{H^{\xi}(\zeta)})$, $\frac{p^{\Delta}(\zeta)}{p^{\xi}(\zeta)} \leq \theta(\frac{H^{\Delta}(\zeta)}{H^{\xi}(\zeta)})$ we see that there is $d \in [s, \xi(s)]$ such that:

$$(p(s)u(s)H^{1-\beta}(s))^{\Delta} = p^{\Delta}(s)u(s)H^{1-\beta}(s) + p(s)u^{\Delta}(s)(H^{\xi}(s))^{1-\beta} + p(s)u(s)(H^{1-\beta}(s))^{\Delta}$$

$$\geq \theta p^{\xi}(s)u^{\xi}(s)H^{\Delta}(s)(H^{\xi}(s))^{-\beta} + \phi p^{\xi}(s)u^{\xi}(s)H^{\Delta}(s)(H^{\xi}(s))^{-\beta} + (1-\beta)p^{\xi}(s)u^{\xi}(s)H^{-\beta}(d)H^{\Delta}(s).$$

Since $H^{\Delta}(s) = h(s) \geq 0$, $d \leq \xi(s)$ and $0 \leq \beta < 1$, we get

$$(p(s)u(s)H^{1-\beta}(s))^{\Delta} \geq \theta p^{\xi}(s)u^{\xi}(s)h(s)(H^{\xi}(s))^{-\beta} + \phi p^{\xi}(s)u^{\xi}(s)h(s)(H^{\xi}(s))^{-\beta} + (1-\beta)p^{\xi}(s)u^{\xi}(s)h(s)(H^{\xi}(s))^{-\beta}.$$

$$= (1-\beta+\theta+\phi)p^{\xi}(s)u^{\xi}(s)h(s)(H^{\xi}(s))^{-\beta}.$$

This gives us that:

$$p^{\xi}(s)u^{\xi}(s)h(s)(H^{\xi}(s))^{-\beta} \leq \frac{1}{1-\beta+\theta+\phi}(p(s)u(s)H^{1-\beta}(s))^{\Delta}.$$

Hence,

$$w^{\xi}(\zeta) = \int_b^{\xi(\zeta)} p^{\xi}(s)u^{\xi}(s)h(s)(H^{\xi}(s))^{-\beta}\Delta s \leq \frac{1}{1-\beta+\theta+\phi}\int_b^{\xi(\zeta)} (p(s)u(s)(H^{1-\beta}(s))^{\Delta}\Delta s$$

$$= \frac{1}{1-\beta+\theta+\phi}p^{\xi}(\zeta)u^{\xi}(\zeta)(H^{\xi}(\zeta))^{1-\beta}. \tag{40}$$

Applying the rules (19) and (25) on time scales, we may observe that there $d \in [\zeta, \xi(\zeta)]$ such that:

$$\Big(-l(\zeta)k(\zeta)v(\zeta)G^q(\zeta))^\Delta\Big) = -\Big((l(\zeta)k(\zeta)v(\zeta))^\Delta G^\xi(\zeta))^q + l(\zeta)k(\zeta)v(\zeta)(G^q(\zeta))^\Delta\Big)$$

$$= -\Big(l^\Delta(\zeta)k(\zeta)v(\zeta)G^\xi(\zeta))^q + l(\zeta)k^\Delta(\zeta)v(\zeta)G^\xi(\zeta))^q + l(\zeta)k(\zeta)v^\Delta(\zeta)G^\xi(\zeta))^q + ql(\zeta)k(\zeta)v(\zeta)G^{q-1}(d)G^\Delta(\zeta)\Big).$$

Since $l^\Delta(\zeta) \geq 0$, $G^\Delta(\zeta) = -r(\zeta)g(\zeta) \leq 0$, $d \geq \zeta$, $q \geq 1, \frac{v^\Delta(\zeta)}{v(\zeta)} \geq \alpha(\frac{G^\Delta(\zeta)}{G^\xi(\zeta)})$, and $\frac{k^\Delta(\zeta)}{k(\zeta)} \geq \gamma(\frac{G^\Delta(\zeta)}{G(\zeta)})$, we have

$$\begin{aligned}\Big(-l(\zeta)k(\zeta)v(\zeta)G^q(\zeta)\Big)^\Delta &\leq \alpha k(\zeta)l(\zeta)v(\zeta)r(\zeta)g(\zeta)(G^\xi(\zeta))^{q-1} + \gamma v(\zeta)l(\zeta)k(\zeta)r(\zeta)g(\zeta)(G^\xi(\zeta))^{q-1} \\ &\quad + ql(\zeta)k(\zeta)v(\zeta)r(\zeta)g(\zeta)(G^\xi(\zeta))^{q-1} \\ &\leq (q+\alpha+\gamma)l(\zeta)v(\zeta)k(\zeta)r(\zeta)g(\zeta)G^{q-1}(\zeta). \end{aligned} \tag{41}$$

It is important to note that $G(\infty) = 0$ and $w(b) = 0$, which we receive. After combining (39), (40), and (41)

$$\begin{aligned}&\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}G^q(\zeta)\Delta\zeta \\ &\leq \frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)r(\zeta)g(\zeta)(H^\xi(\zeta))^{1-\beta}G^{q-1}(\zeta)\Delta\zeta,\end{aligned}$$

or equivalently,

$$\begin{aligned}&\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}G^q(\zeta)\Delta\zeta \\ &\leq \frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\int_b^\infty (l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)h(\zeta))^{\frac{q-1}{q}}(H^\xi(\zeta))^{\frac{-\beta(q-1)}{q}}G^{q-1}(\zeta)) \\ &\times\Big(\frac{(l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta))^{\frac{1}{q}}r(\zeta)g(\zeta)(H^\xi(\zeta))^{\frac{(q-\beta)}{q}}}{h^{\frac{q-1}{q}}(\zeta)}\Big)\Delta\zeta.\end{aligned}$$

The dynamic *Hölder* inequality (28), when applied with indices $q$ and $\frac{q}{q-1}$, allows us to obtain.

$$\begin{aligned}&\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}G^q(\zeta)\Delta\zeta \\ &\leq \frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big(\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}G^q(\zeta)\Delta\zeta\Big)^{\frac{(q-1)}{q}} \\ &\times\Big(\int_b^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)r^q(\zeta)g^q(\zeta)(H^\xi(\zeta))^{(q-\beta)}}{h^{q-1}(\zeta)}\Delta\zeta\Big)^{\frac{1}{q}},\end{aligned}$$

which implies that

$$\begin{aligned}&\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)h(\zeta)(H^\xi(\zeta))^{-\beta}G^q(\zeta)\Delta\zeta \\ &\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q\int_b^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)r^q(\zeta)g^q(\zeta)(H^\xi(\zeta))^{(q-\beta)}}{h^{q-1}(\zeta)}\Delta\zeta.\end{aligned}$$

The proof is now complete.

**Remark 4.** *Assuming $\phi = \alpha = \theta = \gamma = 0$, we may simplify inequality (38) to inequality (16) by setting $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$ and $r(\zeta) = h(\zeta)$.*

**Corollary 5.** *Using relations (23) with Theorem 13 and using $\mathbb{T} = \mathbb{R}$, then inequality (38) becomes:*

$$\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)h(\zeta)H^{-\beta}(\zeta)G^q(\zeta)\Delta\zeta$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \int_b^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)H^{q-\beta}(\zeta)}{h^{q-1}(\zeta)}d\zeta, \tag{42}$$

*where:* $H(\zeta) = \int_b^\zeta h(s)ds$ *and* $G(\zeta) = \int_\zeta^\infty r(s)g(s)ds$.

**Remark 5.** *Assuming $\phi = \alpha = \theta = \gamma = 0$, we may simplify inequality (42) in Corollary 5 with inequality (12) by setting: $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$ and $r(\zeta) = h(\zeta)$ and $b = 0$.*

**Corollary 6.** *Using relations (25) with Theorem 13 and using $\mathbb{T} = f\mathbb{Z}$, then inequality (38) becomes:*

$$\sum_{\zeta=\frac{b}{f}}^\infty l(f\zeta)v(f\zeta)k(f\zeta)p(f\zeta+f)u(f\zeta+f)h(f\zeta)H^{-\beta}(f\zeta+f)G^q(f\zeta)$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \sum_{\zeta=\frac{b}{f}}^\infty \frac{l(f\zeta)v(f\zeta)k(f\zeta)p(f\zeta+f)u(f\zeta+f)r^q(f\zeta)g^q(f\zeta)H^{q-\beta}(f\zeta+f)}{h^{q-1}(f\zeta)}, \tag{43}$$

*where :* $H(\zeta) = f\sum_{s=\frac{b}{f}}^{\frac{\zeta}{f}-1} h(fs)$ *and* $G(\zeta) = f\sum_{s=\frac{\zeta}{f}}^\infty r(fs)g(fs)$.

**Corollary 7.** *In Corollary 6, we just set $f = 1$ for $\mathbb{T} = \mathbb{Z}$. Then inequality (38) becomes*

$$\sum_{\zeta=b}^\infty l(\zeta)v(\zeta)k(\zeta)p(\zeta+1)u(\zeta+1)h(\zeta)H^{-\beta}(\zeta+1)G^q(\zeta)$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \sum_{\zeta=b}^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p(\zeta+1)u(\zeta+1)r^q(\zeta)g^q(\zeta)H^{q-\beta}(\zeta+1)}{h^{q-1}(\zeta)}, \tag{44}$$

*where :* $H(\zeta) = \sum_{s=b}^{\zeta-1} h(s)$ *and* $G(\zeta) = \sum_{s=\zeta}^\infty r(s)g(s)$.

**Remark 6.** *In Corollary 7, if we set $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$, $r(\zeta) = h(\zeta)$, and $b = 1$, then we may take $\phi = \alpha = \theta = \gamma = 0$, so inequality (44) reduces to the inequality (9).*

**Corollary 8.** *Using relations (26) with Theorem 13 and using $\mathbb{T} = \overline{n^{\mathbb{Z}}}$, then inequality (38) becomes*

$$\sum_{\zeta=log_n b}^\infty l(n^\zeta)v(n^\zeta)k(n^\zeta)p(n^{\zeta+1})u(n^{\zeta+1})h(n^\zeta)H^{-\beta}(n^{\zeta+1})G^q(n^\zeta)$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \sum_{\zeta=log_n b}^\infty \frac{l(n^\zeta)v(n^\zeta)k(n^\zeta)p(n^{\zeta+1})u(n^{\zeta+1})r^q(n^\zeta)g^q(n^\zeta)H^{q-\beta}(n^{\zeta+1})}{h^{q-1}(n^\zeta)}, \tag{45}$$

*where :* $H(\zeta) = (n-1)\sum_{s=log_n b}^{(log_n\zeta)-1} n^s h(n^s)$ *and* $G(\zeta) = (n-1)\sum_{s=log_n\zeta}^\infty n^s r(n^s)g(n^s)$.

**Theorem 14.** *Consider $\mathbb{T}$ is a time scale with $b \in [0,\infty)_{\mathbb{T}}$. Inclusion, consider $g,h,l,r,u,v,k,p$ are $\mathbb{R}_1$ rd-continuous functions on $[b,\infty)_{\mathbb{T}}$ that is to say $l$ is non increasing. Furthermore, let us assume that there is $\phi,\theta,\alpha,\gamma \geq 0$ such that $\frac{u^{\Delta}(\zeta)}{u(\zeta)} \leq \phi(\frac{F^{\Delta}(\zeta)}{F^{\xi}(\zeta)})$, $\frac{p^{\Delta}(\zeta)}{p(\zeta)} \leq \theta(\frac{F^{\Delta}(\zeta)}{F^{\xi}(\zeta)})$, $\frac{v^{\Delta}(\zeta)}{v^{\xi}(\zeta)} \leq \alpha(\frac{L^{\Delta}(\zeta)}{L(\zeta)})$ and $\frac{k^{\Delta}(\zeta)}{k^{\xi}(\zeta)} \leq \gamma(\frac{L^{\Delta}(\zeta)}{L(\zeta)})$, where:*

$F(\zeta) = \int_{\zeta}^{\infty} h(s)\Delta s$ *and* $\int_{b}^{\zeta} r(s)g(s)\Delta s, \quad \zeta \in [b,\infty)_{\mathbb{T}}$.

*If* $q1q \geq 1and0 < 10 \leq \beta < 1$, *then :to* $\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)(L^{\xi}(\zeta))^{q}\Delta\zeta$

$\leq \left(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\right)^{q} \int_{b}^{\infty} \frac{l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)r^{q}(\zeta)g^{q}(\zeta)F^{q-\xi}(\zeta)}{h^{q-1}(\zeta)}\Delta\zeta.$

*Proof.* Applying the formula of integration by parts to time scales (22) with

$$w^{\Delta}(\zeta) = p(\zeta)u(\zeta)h(\zeta)F^{-\alpha}(\zeta) \qquad \text{and} \qquad x^{\xi}(\zeta) = l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)(L^{\xi}(\zeta))^{q},$$

we have

$$\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)(L^{\xi}(\zeta))^{q}\Delta\zeta$$

$$= [w(\zeta)l(\zeta)v(\zeta)k(\zeta)L^{q}(\zeta)]_{b}^{\infty} + \int_{b}^{\infty} (-w(\zeta))(l(\zeta)v(\zeta)k(\zeta)L^{q}(\zeta))^{\Delta}\Delta\zeta, \tag{46}$$

where we suppose that

$$w(\zeta) = -\int_{\zeta}^{\infty} p(s)u(s)h(s)F^{-\beta}(s)\Delta s.$$

Applying the rules (21) and (25) on time scales, and the hypothesis $\frac{u^{\Delta}(\zeta)}{u(\zeta)} \leq \phi(\frac{F^{\Delta}(\zeta)}{F^{\xi}(\zeta)})$, $\frac{p^{\Delta}(\zeta)}{p(\zeta)} \leq \theta(\frac{F^{\Delta}(\zeta)}{F^{\xi}(\zeta)})$ we see that there is $d \in [s,\xi(s)]$ such that

$$\begin{aligned}(-p(s)u(s)F^{1-\beta}(s))^{\Delta} &= -\Big(p^{\Delta}(s)u(s)(F^{\xi}(s))^{1-\beta} + p(s)u^{\Delta}(s)(F^{\xi}(s))^{1-\beta} \\ &\quad + p(s)u(s)(F^{1-\beta}(s))^{\Delta}\Big) \\ &\geq -\Big(\theta p(s)u(s)(F^{\xi}(s))^{-\beta}F^{\Delta}(s) + \phi p(s)u(s)(F^{\xi}(s))^{-\beta}F^{\Delta}(s) \\ &\quad + (1-\beta)p(s)u(s)F^{-\beta}(d)F^{\Delta}(s)\Big).\end{aligned}$$

Since $F^{\Delta}(s) = h(s) \leq 0$, $d \geq s$ and $0 \leq \beta < 1$, we get:

$$\begin{aligned}(-p(s)u(s)F^{1-\beta}(s))^{\Delta} &\geq \phi p(s)u(s)h(s)F^{-\beta}(s) + \phi p(s)u(s)h(s)F^{-\beta}(s) \\ &\quad + (1-\beta)p(s)u(s)h(s)F^{-\beta}(s). \\ &= (1-\beta+\theta+\phi)p(s)u(s)h(s)F^{-\beta}(s).\end{aligned}$$

This gives us that:

$$p(s)u(s)h(s)F^{-\beta}(s) \leq \frac{1}{1-\beta+\theta+\phi}(-p(s)u(s)F^{1-\beta}(s))^{\Delta}.$$

Hence

$$-w(\zeta) = \int_{\zeta}^{\infty} p(s)u(s)h(s)F^{-\beta}(s)\Delta s \leq \frac{1}{1-\beta+\theta+\phi}\int_{\zeta}^{\infty} (-p(s)u(s)F^{1-\beta}(s)^{\Delta}\Delta s$$

$$= \frac{1}{1-\beta+\theta+\phi} p(\zeta)u(\zeta)F^{1-\beta}(\zeta). \tag{47}$$

Applying the rules (21) and (25) on time scales, we may observe that there $d \in [\zeta, \xi(\zeta)]$ such that

$$\begin{aligned}\left(l(\zeta)v(\zeta)k(\zeta)L^{q}(\zeta)\right)^{\Delta} &= (l(\zeta)v(\zeta)k(\zeta))^{\Delta}L^{q}(\zeta) + l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)(L^{q}(\zeta))^{\Delta} \\ &= l^{\Delta}(\zeta)v(\zeta)k(\zeta)L^{q}(\zeta) + l^{\beta}(\zeta)k^{\Delta}(\zeta)v(\zeta)L^{q}(\zeta) + l^{\xi}(\zeta)k(\zeta)v^{\Delta}(\zeta)L^{q}(\zeta) \\ &\quad + ql^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)L^{q-1}(d)L^{\Delta}(\zeta).\end{aligned}$$

Since $l^{\Delta}(\zeta) \le 0$, $L^{\Delta}(\zeta) = r(\zeta)g(\zeta) \ge 0$, $d \le \xi(\zeta)$, $q \ge 1, \frac{v^{\Delta}(\zeta)}{v^{\xi}(\zeta)} \le \alpha(\frac{L^{\Delta}(\zeta)}{L(\zeta)})$, and $\frac{k^{\Delta}(\zeta)}{k^{\xi}(\zeta)} \le \gamma(\frac{L^{\Delta}(\zeta)}{L(\zeta)})$, we have

$$\begin{aligned}\left(l(\zeta)v(\zeta)k(\zeta)L^{q}(\zeta)\right)^{\Delta} &\le \alpha l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)r(\zeta)g(\zeta)L^{q-1}(\zeta) + \gamma l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)r(\zeta)g(\zeta)L^{q-1}(\zeta) \\ &\quad + ql^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)r(\zeta)g(\zeta)(L^{\xi}(\zeta))^{q-1} \\ &\le (q+\alpha+\gamma)l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)r(\zeta)g(\zeta)(L^{\xi}(\zeta))^{q-1}.\end{aligned} \tag{48}$$

It is important to note that $L(b) = 0$ and $w(\infty) = 0$, which we receive. After combining (47), (48) and (491)

$$\begin{aligned}&\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)(L^{\xi}(\zeta))^{q}\Delta\zeta \\ &\le \frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)r(\zeta)g(\zeta)F^{1-\beta}(\zeta)(L^{\xi}(\zeta))^{q-1}\Delta\zeta,\end{aligned}$$

or equivalently,

$$\begin{aligned}&\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)(L^{\xi}(\zeta))^{q}\Delta\zeta \\ &\le \frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\int_{b}^{\infty} (l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta))^{\frac{q-1}{q}}F^{\frac{-\beta(q-1)}{q}}(\zeta)(L^{\xi}(\zeta))^{q-1}) \\ &\times\Big(\frac{(l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta))^{\frac{1}{q}}r(\zeta)g(\zeta)F^{\frac{(q-\beta)}{q}}}{h^{\frac{q-1}{q}}(\zeta)}\Big)\Delta\zeta.\end{aligned}$$

The dynamic *Hölder* inequality (28), when applied with indices $q$ and $\frac{q}{q-1}$, allows us to obtain.

$$\begin{aligned}&\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)(L^{\xi}(\zeta))^{q}\Delta\zeta \\ &\le \frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big(\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)(L^{\xi}(\zeta))^{q}\Delta\zeta\Big)^{\frac{(q-1)}{q}} \\ &\times\Big(\int_{b}^{\infty} \frac{l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)r^{q}(\zeta)g^{q}(\zeta)F^{q-\beta}(\zeta)}{h^{q-1}(\zeta)}\Delta\zeta\Big)^{\frac{1}{q}},\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{b}^{\infty} l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)(L^{\xi}(\zeta))^{q}\Delta\zeta \\ &\le \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^{q}\int_{b}^{\infty} \frac{l^{\xi}(\zeta)v^{\xi}(\zeta)k^{\xi}(\zeta)p(\zeta)u(\zeta)r^{q}(\zeta)g^{q}(\zeta)F^{q-\beta}(\zeta)}{h^{q-1}(\zeta)}\Delta\zeta.\end{aligned}$$

The proof is now complete.

**Remark 7.** *Assuming $\phi = \alpha = \theta = \gamma = 0$, we may simplify inequality (46) to inequality (17) with theorem 14 by setting $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$ and $r(\zeta) = h(\zeta)$.*

**Corollary 9.** *Using relations (23) in the Theorem 14 and using $\mathbb{T} = \mathbb{R}$, then inequality (46) becomes:*

$$\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)L^q(\zeta)d\zeta$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \int_b^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)F^{q-\beta}(\zeta)}{h^{q-1}(\zeta)}d\zeta, \tag{49}$$

*where: $F(\zeta) = \int_\zeta^\infty h(s)ds$ and $L(\zeta) = \int_b^\zeta r(s)g(s)ds$.*

**Remark 8.** *In Corollary 9, if we set $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$, $r(\zeta) = h(\zeta)$, and $b = 0$, then we take $\phi = \alpha = \theta = \gamma = 0$, so inequality (49) reduces to:*

$$\int_0^\infty \frac{h(\zeta)(\int_0^\zeta h(s)g(s)ds)^q}{(\int_\zeta^\infty h(s)ds)^\beta}d\zeta \leq \Big(\frac{q}{1-\beta}\Big)^q \int_0^\infty h(\zeta)g^q(\zeta)\Big(\int_\zeta^\infty h(s)ds\Big)^{q-\beta}d\zeta, \tag{50}$$

*According to this interpretation, the discrete inequality (10) can be understood differently.*

**Corollary 10.** *Using relations (25) with Theorem 14 and using $\mathbb{T} = f\mathbb{Z}$, then inequality (46) becomes:*

$$\sum_{\zeta=\frac{b}{f}}^\infty l(f\zeta+f)v(f\zeta+f)k(f\zeta+f)p(f\zeta)u(f\zeta)h(f\zeta)F^{-\beta}(f\zeta)L^q(f\zeta+f)$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \sum_{\zeta=\frac{b}{f}}^\infty \frac{l(f\zeta+f)v(f\zeta+f)k(f\zeta+f)p(f\zeta)u(f\zeta)r^q(f\zeta)g^q(f\zeta)F^{q-\beta}(f\zeta)}{h^{q-1}(f\zeta)}, \tag{51}$$

*where*

*$F(\zeta) = f\sum_{s=\frac{\zeta}{f}}^\infty h(fs)$ and $L(\zeta) = f\sum_{s=\frac{b}{f}}^{\frac{\zeta}{f}-1} r(fs)g(fs)$.*

**Corollary 11.** *In Corollary 10, we just set $f = 1$ for $\mathbb{T} = \mathbb{Z}$. Then inequality (46) becomes:*

$$\sum_{\zeta=b}^\infty l(\zeta+1)v(\zeta+1)k(\zeta+1)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta+1)L^q(\zeta+1)$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \sum_{\zeta=b}^\infty \frac{l(\zeta+1)v(\zeta+1)k(\zeta+1)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)F^{q-\beta}(\zeta)}{h^{q-1}(\zeta)}, \tag{52}$$

*where : $F(\zeta) = \sum_{s=b}^\infty h(s)$ and $L(\zeta) = \sum_{s=b}^{\zeta-1} r(s)g(s)$.*

**Corollary 12.** *Using relations (26) with Theorem 14 and using $\mathbb{T} = \overline{n^{\mathbb{Z}}}$, then inequality (46) becomes:*

$$\sum_{\zeta=log_n b}^\infty l(n^{\zeta+1})v(n^{\zeta+1})k(n^{\zeta+1})p(n^\zeta)u(n^\zeta)h(n^\zeta)F^{-\beta}(n^\zeta)L^q(n^{\zeta+1})$$
$$\leq \Big(\frac{q+\alpha+\gamma}{1-\beta+\theta+\phi}\Big)^q \sum_{\zeta=log_n b}^\infty \frac{l(n^{\zeta+1})v(n^{\zeta+1})k(n^{\zeta+1})p(n^\zeta)u(n^\zeta)r^q(n^\zeta)g^q(n^\zeta)F^{q-\beta}(n^\zeta)}{h^{q-1}(n^\zeta)}, \tag{53}$$

*where : $F(\zeta) = (n-1)\sum_{s=log_n\zeta}^\infty n^s h(n^s)$ and $L(\zeta) = (n-1)\sum_{s=log_n b}^{(log_n\zeta)-1} n^s r(n^s)g(n^s)$.*

**Theorem 15.** *Consider $\mathbb{T}$ is a time scale with $b \in [0,\infty)_{\mathbb{T}}$. Inclusion, consider $g, h, l, r, u, p, v, k$ are $\mathbb{R}_1$ rd-continuous functions on $[b,\infty)_{\mathbb{T}}$ that is to say $l$ is non decreasing. Furthermore, let us assume that there is $\phi, \theta, \alpha, \gamma \geq 0$ such that $\frac{u^{\Delta}(\zeta)}{u^{\xi}(\zeta)} \leq \phi(\frac{F^{\Delta}(\zeta)}{F(\zeta)})$, $\frac{p^{\Delta}(\zeta)}{p^{\xi}(\zeta)} \leq \theta(\frac{F^{\Delta}(\zeta)}{F(\zeta)})$, $\frac{v^{\Delta}(\zeta)}{v(\zeta)} \leq \alpha(\frac{G^{\Delta}(\zeta)}{G^{\xi}(\zeta)})$ and $\frac{k^{\Delta}(\zeta)}{k^{\xi}(\zeta)} \leq \gamma(\frac{G^{\Delta}(\zeta)}{G(\zeta)})$, where:*
*$F(\zeta) = \int_{\zeta}^{\infty} h(s)\Delta s$ and $G(\zeta) = \int_{\zeta}^{\infty} r(s)g(s)\Delta s$, $\quad \zeta \in [b,\infty)_{\mathbb{T}}$.*
*If $q \geq 1$ and $\beta > \phi + 1$, then*

$$\int_{b}^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)F^{-\beta}(\zeta)G^{q}(\zeta)\Delta\zeta$$
$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^{q} \int_{b}^{\infty} \frac{l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)r^{q}(\zeta)g^{q}(\zeta)F^{\beta(q-1)}(\zeta)}{h^{q-1}(\zeta)(F^{\xi}(\zeta))^{q(\beta-1)}}\Delta\zeta. \tag{54}$$

*Proof.* Applying the formula of integration by parts to time scales (22) with:

$$\int_{b}^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)F^{-\beta}(\zeta)G^{q}(\zeta)\Delta\zeta$$
$$= [w(\zeta)l(\zeta)v(\zeta)k(\zeta)G^{q}(\zeta)]_{b}^{\infty} + \int_{b}^{\infty} w^{\xi}(\zeta)(-l(\zeta)v(\zeta)k(\zeta)G^{q}(\zeta))^{\Delta}\Delta\zeta, \tag{55}$$

where we suppose that:

$$w(\zeta) = \int_{b}^{\zeta} p^{\xi}(s)u^{\xi}(s)h(s)F^{-\beta}(s)\Delta s.$$

Applying the rules (21) and (27) on time scales, and the hypothesis $\frac{u^{\Delta}(\zeta)}{u^{\xi}(\zeta)} \geq \phi(\frac{F^{\Delta}(\zeta)}{F(\zeta)})$, $\frac{p^{\Delta}(\zeta)}{p^{\xi}(\zeta)} \geq \theta(\frac{F^{\Delta}(\zeta)}{F(\zeta)})$ we see that there is $d \in [s,\xi(s)]$ such that:

$$(p(s)u(s)F^{1-\beta}(s))^{\Delta} = p^{\Delta}(s)u^{\xi}(s)F^{1-\beta}(s) + p^{\xi}(s)u^{\Delta}(s)(F^{\xi}(s))^{1-\beta} + p^{\xi}(s)u^{\xi}(s)(F^{1-\beta}(s))^{\Delta}$$

$$\geq \theta p^{\xi}(s)u^{\xi}(s)F^{-\beta}(s)F^{\Delta}(s) + \phi p^{\xi}(s)u^{\xi}(s)F^{-\beta}(s)F^{\Delta}(s)$$
$$+(1-\beta)p^{\xi}(s)u^{\xi}(s)F^{-\beta}(d)F^{\Delta}(s).$$

Since $F^{\Delta}(s) = -h(s) \leq 0$, $d \geq s$ and $\beta > 1$, we get:

$$(p(s)u(s)F^{1-\beta}(s))^{\Delta} \geq -\theta p^{\xi}(s)u^{\xi}(s)h(s)F^{-\beta}(s) - \phi p^{\xi}(s)u^{\xi}(s)h(s)F^{-\beta}(s)$$
$$+(\beta-1)p^{\xi}(s)u^{\xi}(s)h(s)F^{-\beta}(s).$$

$$= (\beta-\theta-\phi-1)p^{\xi}(s)u^{\xi}(s)h(s)F^{-\beta}(s).$$

This gives us that

$$p^{\xi}(s)u^{\xi}(s)h(s)F^{-\beta}(s) \leq \frac{1}{\beta-\theta-\phi-1}(p(s)u(s)F^{1-\beta}(s)^{\Delta}.$$

Hence

$$w^{\xi}(\zeta) = \int_{b}^{\xi(\zeta)} p^{\xi}(s)u^{\xi}(s)h(s)F^{-\beta}(s)\Delta s \leq \frac{1}{\beta-\theta-\phi-1}\int_{b}^{\xi(\zeta)} (p(s)u(s)(F^{1-\beta}(s))^{\Delta}\Delta s$$

$$= \frac{1}{\beta-\theta-\phi-1}\Big(p^{\xi}(\zeta)u^{\xi}(\zeta)(F^{\xi}(\zeta))^{1-\beta} - p(b)u(b)(F(b))^{1-\beta}\Big)$$

$$\leq \frac{1}{\beta-\theta-\phi-1}p^{\xi}(\zeta)u^{\xi}(\zeta)(F^{\xi}(\zeta))^{1-\beta}. \tag{56}$$

Applying the rules (21) and (27) on time scales, we may observe that there $d \in [\zeta, \xi(\zeta)]$ such that:

$$\Big(-l(\zeta)v(\zeta)k(\zeta)G^{q}(\zeta))^{\Delta}\Big) = -\Big((l(\zeta)v(\zeta)k(\zeta))^{\Delta}G^{\xi}(\zeta))^{q} + l(\zeta)v(\zeta)k(\zeta)(G^{q}(\zeta))^{\Delta}\Big)$$

$$= -\Big(l^{\Delta}(\zeta)v(\zeta)k(\zeta)G^{\xi}(\zeta))^{q} + l(\zeta)v^{\Delta}(\zeta)k(\zeta)G^{\xi}(\zeta))^{q} + l(\zeta)v(\zeta)k^{\Delta}(\zeta)G^{\xi}(\zeta))^{q} + ql(\zeta)v(\zeta)k(\zeta)G^{q-1}(d)G^{\Delta}(\zeta)\Big).$$

Since $l^{\Delta}(\zeta) \geq 0$, $G^{\Delta}(\zeta) = -r(\zeta)g(\zeta) \leq 0$, $d \geq \zeta$, $q \geq 1, \frac{v^{\Delta}(\zeta)}{v(\zeta)} \geq \alpha(\frac{G^{\Delta}(\zeta)}{G^{\xi}(\zeta)})$, and $\frac{k^{\Delta}(\zeta)}{k(\zeta)} \geq \gamma(\frac{G^{\Delta}(\zeta)}{G^{\xi}(\zeta)})$, we have:

$$\Big(-l(\zeta)v(\zeta)k(\zeta)G^{q}(\zeta)\Big)^{\Delta} \leq \alpha l(\zeta)v(\zeta)k(\zeta)r(\zeta)g(\zeta)(G^{\xi}(\zeta))^{q-1} + \gamma l(\zeta)v(\zeta)k(\zeta)r(\zeta)g(\zeta)(G^{\xi}(\zeta))^{q-1} + ql(\zeta)v(\zeta)k(\zeta)r(\zeta)g(\zeta)(G^{\xi}(\zeta))^{q-1}$$

$$\leq (q+\alpha+\gamma)l(\zeta)v(\zeta)k(\zeta)r(\zeta)g(\zeta)G^{q-1}(\zeta). \tag{57}$$

It is important to note that $G(\infty) = 0$ and $w(b) = 0$, which we receive. After combining (54), (55) and (56):

$$\int_b^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)F^{-\beta}(\zeta)G^{q}(\zeta)\Delta\zeta$$

$$\leq \frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\int_b^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)r(\zeta)g(\zeta)(F^{\xi}(\zeta))^{1-\beta}G^{q-1}(\zeta)\Delta\zeta,$$

or equivalently,

$$\int_b^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)F^{-\beta}(\zeta)G^{q}(\zeta)\Delta\zeta$$

$$\leq \frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\int_b^{\infty}(l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta))^{\frac{q-1}{q}}F^{\frac{-\beta(q-1)}{q}}(\zeta)G^{q-1}(\zeta))$$

$$\times\Big(\frac{(l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta))^{\frac{1}{q}}r(\zeta)g(\zeta)F^{\frac{\beta(q-1)}{q}}}{h^{\frac{q-1}{q}}(\zeta)(F^{\xi}(\zeta))^{\beta-1}}\Big)\Delta\zeta.$$

The dynamic *Hölder* inequality (28), when applied with indices $q$ and $\frac{q}{q-1}$, allows us to obtain.

$$\int_b^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)F^{-\beta}(\zeta)G^{q}(\zeta)\Delta\zeta$$

$$\leq \frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big(\int_b^{\infty} l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)h(\zeta)F^{-\beta}(\zeta)G^{q}(\zeta)\Delta\zeta\Big)^{\frac{(q-1)}{q}}$$

$$\times\Big(\int_b^{\infty}\frac{l(\zeta)v(\zeta)k(\zeta)p^{\xi}(\zeta)u^{\xi}(\zeta)r^{q}(\zeta)g^{q}(\zeta)F^{\beta(q-1)}(\zeta)}{h^{q-1}(\zeta)(F^{\xi}(\zeta))^{q(\beta-1)}}\Delta\zeta\Big)^{\frac{1}{q}},$$

which implies that

$$\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)h(\zeta)F^{-\beta}(\zeta)G^q(\zeta)\Delta\zeta$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \int_b^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p^\xi(\zeta)u^\xi(\zeta)r^q(\zeta)g^q(\zeta)F^{\beta(q-1)}(\zeta)}{h^{q-1}(\zeta)(F^\xi(\zeta))^{q(\beta-1)}}\Delta\zeta.$$

The proof is now complete.

**Remark 9.** *Assuming $\phi = \alpha = \theta = \gamma = 0$, we may simplify inequality* (54) *to inequality* (18) *with Theorem 15 by setting $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$ and $r(\zeta) = h(\zeta)$.*

**Corollary 13.** *Using relations* (23) *with Theorem 15 and using $\mathbb{T} = \mathbb{R}$, then inequality* (54) *becomes:*

$$\int_b^\infty l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)h(\zeta)F^{-\beta}(\zeta)G^q(\zeta)\Delta\zeta$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \int_b^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p(\zeta)u(\zeta)r^q(\zeta)g^q(\zeta)F^{q-\beta}(\zeta)}{h^{q-1}(\zeta)}d\zeta, \tag{58}$$

*where*

$F(\zeta) = \int_\zeta^\infty h(s)ds$ *and* $G(\zeta) = \int_\zeta^\infty r(s)g(s)ds.$

**Remark 10.** *In Corollary 13, if we set $l(\zeta) = v(\zeta) = k(\zeta) = p(\zeta) = u(\zeta) = 1$, $r(\zeta) = h(\zeta)$, and $b = 0$, then we take $\phi = \alpha = \theta = \gamma = 0$, so inequality* (58) *reduces to:*

$$\int_0^\infty \frac{h(\zeta)(\int_\zeta^\infty h(s)g(s)ds)^q}{(\int_\zeta^\infty h(s)ds)^\beta}d\zeta \leq \Big(\frac{q}{\beta-1}\Big)^q \int_0^\infty h(\zeta)g^q(\zeta)\Big(\int_\zeta^\infty h(s)g(s)ds\Big)^{q-\beta}d\zeta, \tag{59}$$

*This is the discrete inequality (13), but in continuous form.*

**Corollary 14.** *Using relations* (25) *in the theorem 15 and using $\mathbb{T} = f\mathbb{Z}$, then inequality* (53) *becomes:*

$$\sum_{\zeta=\frac{b}{f}}^\infty l(f\zeta)v(f\zeta)k(f\zeta)p(f\zeta+f)u(f\zeta+f)h(f\zeta)F^{-\beta}(f\zeta)G^q(f\zeta)$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \sum_{\zeta=\frac{b}{f}}^\infty \frac{l(f\zeta)v(f\zeta)k(f\zeta)p(f\zeta+f)u(f\zeta+f)r^q(f\zeta)g^q(f\zeta)F^{\beta(q-1)}(f\zeta)}{h^{q-1}(\zeta)F^{q(\beta-1)}(f\zeta+f)}, \tag{60}$$

*where:*
$F(\zeta) = f\sum_{s=\frac{\zeta}{f}}^\infty h(fs)$ *and* $G(\zeta) = f\sum_{s=\frac{\zeta}{f}}^\infty r(fs)g(fs).$

**Corollary 15.** *In Corollary* 14*, we just set $f = 1$ for $\mathbb{T} = \mathbb{Z}$. Then inequality* (53) *becomes:*

$$\sum_{\zeta=b}^\infty l(\zeta)v(\zeta)k(\zeta)p(\zeta+1)u(\zeta+1)h(\zeta)F^{-\beta}(\zeta)G^q(\zeta)$$

$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^q \sum_{\zeta=b}^\infty \frac{l(\zeta)v(\zeta)k(\zeta)p(\zeta+1)u(\zeta+1)r^q(\zeta)g^q(\zeta)F^{\beta(q-1)}(\zeta)}{h^{q-1}(\zeta)F^{q(\beta-1)}(\zeta+1)}, \tag{61}$$

*where*
$F(\zeta) = \sum_{s=\zeta}^\infty h(s)$ *and* $G(\zeta) = \sum_{s=\zeta}^\infty r(s)g(s).$

**Corollary 16.** *Using relations* (26) *with Theorem 15 and using* $\mathbb{T} = \overline{n^{\mathbb{Z}}}$*, then inequality* (53) *becomes:*

$$\sum_{\zeta=log_n b}^{\infty} l(n^{\zeta})v(n^{\zeta})k(n^{\zeta})p(n^{\zeta+1})u(n^{\zeta+1})h(n^{\zeta})F^{-\beta}(n^{\zeta})G^{q}(n^{\zeta})$$
$$\leq \Big(\frac{q+\alpha+\gamma}{\beta-\theta-\phi-1}\Big)^{q} \sum_{\zeta=log_n b}^{\infty} \frac{l(n^{\zeta})v(n^{\zeta})k(n^{\zeta})p(n^{\zeta+1})u(n^{\zeta+1})r^{q}(n^{\zeta})g^{q}(n^{\zeta})F^{\beta(q-1)}(n^{\zeta})}{h^{q-1}(n^{\zeta})F^{q(\beta-1)}(n^{\zeta+1})}, \tag{62}$$

*where*
$F(\zeta) = (n-1)\sum_{s=log_n\zeta}^{\infty} n^{s}h(n^{s})$ *and* $G(\zeta) = (n-1)\sum_{s=log_n\zeta}^{\infty} n^{s}r(n^{s})g(n^{s})$.

## 4. DISCUSSION AND CONCLUSION

We proposed some new dynamic Hardy-type inequalities in this work by using the time scales' version of *Hölder* inequality, the integration by parts method, and Keller's chain rule. Read the remarks and corollaries that come after each of our main results to learn more about how the inequalities we proved apply to other dynamic inequalities that have already been written about. We applied the theorems to a variety of time scales, including $\mathbb{R}$, $f\mathbb{Z}$, $\overline{n^{\mathbb{Z}}}$, and $\mathbb{Z}$ as a subcase of $f\mathbb{Z}$, in order to demonstrate each type of inequality. By combining the strengths of both discrete and continuous analysis, these inequalities open up new avenues for research into the temporal features of dynamic processes. Applying these inequalities to more complicated systems with jumps and hybrid dynamics is just the beginning of their extensive range of possible uses. Extended and generalised dynamic Hardy-type inequalities on time scales are still fundamental to the development of dynamic equations and mathematical analysis in this area of study. It is possible that future study will investigate a variety of generalisations and modifications of the dynamic Hardy inequality by using the findings presented in this paper. The results of this paper will help us understand more about many mathematical problems, especially functional analysis. This study will present the mathematical framework for the Hardy type inequality and show its importance in understanding fundamental functions and their interconnections.

### Conflicts Of Interest

The authors declare no conflicts of interest.


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