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Research Article

Fuzzy Soc–Small Two-Absorbing Modules and Related Concepts

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ABSTRACT

This study presents the fuzzy socle small two-Absorbing module and compares it to other fuzzy modules, including fuzzy small prime, fuzzy small two-Absorbing, and fuzzy two-Absorbing. Numerous aspects found during the investigation and discussion supported the new ideas, which fall within the fuzzy hollow, fuzzy chain, and fuzzy multiplication module classes. Discuss the connection between this concept and the fuzzy socle small two-Absorbing ideal. To create a new fuzzy socle small two-Absorbing, these findings are required.

1. INTRODUCTION

The definition of a fuzzy subset was initially given by Zadeh [1] in 1965. Rahman and Saikia [13] developed the idea of a fuzzy small sub-module. Assume that X is a fuzzy module of a T-module G and that P is a fuzzy sub-module of X. If P means that $P + S \neq X$ for every suitable fuzzy sub-module S of X, then P is a fuzzy tiny sub-module of X. Hatam and Wafaa coined the idea of a fuzzy small prime sub-module [8]. If and only if fuzzy singleton is present, then fuzzy sub-module P of fuzzy module X of T-module G is considered legitimate a_s *of* T and $x_v \subseteq X$,$\forall s,v \in [0,1]$, with $\langle x_v \rangle \ll X$ and $a_s x_v \subseteq$ P, implies either $x_v \subseteq$ P *or* $a_s \subseteq$ (P:X). Additionally, the fuzzy Two-Absorbing sub-module was proposed by Wafaa and Hatam [3]: Assume A *fuzzy sub-modul*e A T-module G's fuzzy module is U *of* X if $q_s r_b m_t \subseteq U$, with each $t, b, s \in [0;\ 1]$, either $q_s m_t \subseteq U$ or $r_b m_t \subseteq U$, or $q_s\, r_b \subseteq [U : X]$ if m_t is fuzzy. Wafaa and Hatam [8] also proposed a fuzzy small Two-Absorbing sub-module in (2018): Assume X is a *fuzzy module* and a *fuzzy sub-module* K *of* X is named fuzzy small Two-Absorbing if $q_s\, r_n\, m_t \subseteq U$,with q_s, r_n *fuzzy singleton of* T and $< m_t > \ll X$, thence that choice of $q_s m_t \subseteq P$ or $r_n m_t \subseteq P$ or $q_s\, r_n \subseteq [P : X]$ for each $t, s, n \in [0;\ 1]$.
The fuzzy socle small Two-Absorbing module is a fuzzy extension of the fuzzy small prime module and fuzzy small Two-Absorbing module concepts. This article consists of two sections. We list a number of key words and traits that we will require in the first section. Part two examines the fuzzy Socle small Two-Absorbing module's outputs, outcomes, and a number of important aspects.Note that: The notations fzy ideal, fzy sub-module, fzy singleton, fzy Socle and fzy module represent the fuzzy ideal, fuzzy sub-module, fuzzy singleton, fuzzy Socle and fuzzy module.

2. PRELIMINARIES

This part covers the many foundational ideas and any preconditions they might have for the next section.

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Definition 2.1[1]**:**
Let I be an interval [0,1] of the real line (real number) and S be a non-empty set. A function from S into I is a fzy set A in S (a fzy subset of S).

Definition 2.2 [5]:
Assume $x_u: S \to I$ be a fzy set in S, defined by: $x \in S, u \in I$ so
$x_u(n) = \begin{cases} u \; if \; x = n \\ 0 \; if \; x \neq n \end{cases}$, x_u is named fzy singleton in S.
If $x = 0$ and $u = 1$, then $0_1(n) = \begin{cases} 1 \; if \; n = 0 \\ 0 \; if \; n \neq 0 \end{cases}$

Definition 2.3 [2]
Assuming that P is a fzy set in S, the set $P_t = \{x \in S; P(x) \geq t\}$ For all $t \in I$, is named a level subset of P, if P is a fzy set in S. In the ordinary sense, P_t is a subset of S; remember that**.**

Definition 2.5 [12]
Suppose D is a T-module. A T-module's fzy module is a fzy set X of D. if
1- $X\,(n - m) \geq min\,(X\,(n), X\,(m), \forall\, n, m \in D\}$.
2- $X(rn) \geq X\,(n)\, \forall\, n \in G \; and \; r \in T$.
3- $X(0) = 1$.

Definition 2.6 [9]
Let X and P be two fzy $modules\ of$ T-module D If $P \subseteq X$, then P is a fzy sub-module $of\ X$.

Proposition 2.7 [2]
Make a T-module of D's fzy set P. The level subset $P_l, l \in I$ then is a sub-module of D if P $is\ a$ fzy sub-module $of\ X$ where X is an fzy $module\ of\ a$ T-MODULE D

Definition 2.8 [12]
A fzy $idea$ of a rin D T is defined as a fzy $-subset\ K$ if $\forall n, m \in T$:
1) $K(n - m) \geq min\,\{K(n), K(m)\}$
2) $K(nm) \geq max\,\{K(n), K(m)\}$.
Definition 2.9 [12]
Two fzy sub-modules $of\ a$ fzy $module\ X$ should be P and S. $(P : S)$ represents the residual quotient of P and S, which is the fzy $subst\ of$ T defined as:
$(P : S)(r) = sup\{l \in [0{:}\,1]{:}\, r_l S \subseteq P\}, \forall\, r \in T.That\ (P : S) = \{r_t{:}\, r_t S \subseteq P; r_t$ is a fzy -singleton of T}.
If S $=< x_k >$,$then\ (P :< x_k >) = \{r_t{:}\, r_t\, x_k \subseteq P{:}\, r_t\ is\ a\ Fzy - singleton\ of\ T\}$.

Proposition 2.10 [10]
Assume that P and S are two fzy sub-modules of fzy module X of a T-module D. $(P : S)$ is therefore a fzy ideal of T.

Definition 2.11 [12]
Assume that P and Q are two fzy sub-modules of D, which is a T-module. The definition of an addition is $P + Q$. by:
$(P + Q)(u) = sup\,\{inf\,\{P(e), Q(g)\}\ with\ u = e + g, \forall\ u, e, g \in G\}$. Moreover, $P + Q$ is a fzy sub-module of a T-module D.

Corollary 2.12 [9]
For every fzy singleton , $t_k$of T, $t_k\, n_l = (rx)\lambda$,, where λ = min{l, k}, if X is a fzy module of a T-module D and $n_l \subseteq$ X.

Definition 2.13 [7]
If and only if X contains only petty fzy sub-modules, then X is fzy simple (in fact, X is fzy simple iff X has just 0_1 and itself). Let X be a fzy module of a T-module D.

Definition 2.14 [10]

When X is the sum of its simple fzy sub-modules, it is referred to as semi-simple. Moreover, $Fzy - Soc(X)$ represents the Fzy Socle of X, which is the sum of the simple fzy sub-module of X. When X equals $Fzy - Soc\ (X)$, it is regarded as semi-simple.

Definition 2.15 [6]

If X is a fzy module of a T-module D and $Fzy - annX = 0_1$, then X is fzy faithful module. Where $Fzy - annX = \{t_k : t_k\ u_l = 0_1;\ \forall\ u_l \subseteq X\ and\ t_k\ be\ a$ fzy $singleton\ of$ T$\}$

Definition 2.16 [7]

Make X a fzy module of D as a T-module. X is called a multiplication fzy module iff there is a fzy $ideal$ K of T such that $A = KX$.

Proposition 2.17 [7]

A $fzy - module\ X\ of\ a\ T - module\ D$ is multiplication if and only if every non-empty fzy sub-module $P\ of\ X$ such that $P = (P : X)X$.

Lemma 2.18 [16]

As the T-module, make X the fzy module of D. $Fzy - Soc(T)X = Fzy - Soc(X)$ if D is a faithful multiplication T-module in this case.

Definition 1.19 [7]

If $x_s \subseteq X$ such that $g_z \subseteq X$, expressed as $g_z = e_l\ x_s$ for some fzy singleton e_l of T, then a fzy module X of a T-module D is referred to as a Fzy cyclic module, where $z, l, s \in I$ in this case, write $X =< x_s >$ to denote the Fzy- cyclic module generated by x_s.

Definition 2.20 [4]

X is referred to as a finitely produced Fzy module if X is a fzy module of a T-module D. if there exists $x_1, x_2, x_3, \ldots \subseteq X$ such that $= \{a_1(x_1)v_1 + a_2(x_2)v_2 + \cdots + a_n(x_n)v_n\}$, where $a_i \in T\ and\ a(x)_v = (ax)_v, \forall v \in I$.

Where $(ax)_v(g) = \begin{cases} v & if\ g = ax \\ 0 & o.w. \end{cases}$

Definition 2.21 [4]

A Fzy cancellation module is a fzy module of a T-module D if KX=LX, where K and L are Fzy-ideals of T, and then K=L.

Proposition 2.22 [4]

A fzy module X is named cancellation module if it is a Fzy faithful finitely generated multiplication of a T-module D.

Definition 2.23[13]

A Fzy small ideal of T is said to have H(0) = 1 if it is a Fzy small sub-module of a Fzy module X of a T-module T.

Definition 2.24 [14]

A fzy module X of a T-module D, with $X \neq 0_1$ is called a Fzy-hollow module if for every Fzy sub-module A with $A \neq X$, implies that $A \ll X$.

Definition 2.25 [15]:

Let X be a fzy module of a T-module D, then X is called a Fzy chained module if for each fzy sub-modules A and B of X then either $A \subseteq B$ or $B \subseteq A$.

Proposition 2.26 [16]

If X is a fzy module of D and D is a faithful multiplication of a T-module, then $Fzy - Soc(X) = F - Soc(T)X$.

Remark 2.27 [16]

If a faithful multiplication fzy module X of a T-module D has a fzy sub-module P, then $(P{:}X) + Fzy - Soc(T) \subseteq (P + F - Soc(X){:}X)$

3. Fzy Socle Small Two Absorbing Modules

Definition 3.1:

A fzy module X of a T-module D is called a Fzy Socle small **T**wo **A**bsorbing module (shortly Fzy Soc-STA) if 0_1 is a Fzy Soc-STA sub-module; that is if $s_a y_b x_v \subseteq 0_1$ for fzy singletons s_a, y_b of T and $< x_v > \ll X$ where a, b, v ∈I, then either $s_a x_v \subseteq 0_1 + fzy - Soc(X)$ or $y_b x_v \subseteq 0_1 + fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + fzy - Soc(X): X)$.

Proposition 3.2:

Let X be a fzy module of a T-module D, if X is Fzy small prime module, then X is Fzy Soc-STA module.

Proof:

Let $s_a y_b x_v \subseteq 0_1$ for fzy singletons s_a, y_b of T and $< x_v > \ll X$ where a, b, v ∈I, then $y_b x_v \subseteq 0_1$ or $s_a \subseteq (0_1: X)$, hence $y_b x_v \subseteq 0_1 + fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + fzy - Soc(X): X)$. Thus, X is a fzy Soc-STA module.

Remark 3.3:

The converse of proposition 3.2, is not true in general, for example:

Let $X: Z_{12} \rightarrow I$ such that $X(g) = \begin{cases} 1 \; if \; g \in Z_{12} \\ 0 \quad o.w \end{cases}$

It is clear X is a fzy module of Z_{12} as Z-module.

$$0_1(g) = \begin{cases} 1 \; if g = 0 \\ 0 \; if g \neq 0 \end{cases}$$

$$fzy - Soc(X)(g) = \begin{cases} \frac{1}{2} \; if \; g \in (2) \\ 0 \; if \; g \notin (\bar{2}) \end{cases}$$

$$\big(0_1 + fzy - Soc(X)\big)(g) = \begin{cases} 1 \; if \mathrm{g} \in (\bar{2}) \\ 0 \; if \; g \notin (\bar{2}) \end{cases}$$

$$(0_1 + fzy - Soc(X): X)(g) = \begin{cases} \frac{1}{3} \; if \; g \in 2Z \\ 0 \; if \; g \notin 2Z \end{cases}$$

$$(0_1: X)(g) = \begin{cases} \frac{1}{4} \; if \; g \in 12Z \\ 0 \; if \; g \notin 12Z \end{cases}$$

X is Fzy Soc-STA module since $2_{\frac{1}{4}} \cdot 1_{\frac{1}{3}} \cdot \bar{6}_{\frac{1}{6}} = 0_{\frac{1}{6}} \subseteq 0_1$ and $< \bar{6}_{\frac{1}{6}} > \ll X$, then $2_{\frac{1}{4}} \cdot \bar{6}_{\frac{1}{6}} = 0_{\frac{1}{6}} \subseteq 0_1 + fzy - Soc(X)$ or $1_{\frac{1}{3}} \cdot \bar{6}_{\frac{1}{6}} = 6_{\frac{1}{6}} \subseteq 0_1 + fzy - Soc(X)$ or $2_{\frac{1}{4}} \cdot 1_{\frac{1}{3}} = 2_{\frac{1}{4}} \in (0_1 + fzy - Soc(X): X)$ since $\big(0_1 + fzy - Soc(X)\big)(0) = 1 > \frac{1}{6}$, $0_1 + fzy - Soc(X)(6) = 1 > \frac{1}{6}$ and $(0_1 + fzy - Soc(X): X)(2) = \frac{1}{3} > \frac{1}{4}$, but X is not a Fzy small prime module since $2_{\frac{1}{6}} \cdot \bar{6}_{\frac{1}{3}} = 0_{\frac{1}{6}} \subseteq 0_1$ with $< \bar{6}_{\frac{1}{6}} > \ll X$, but $6_{\frac{1}{3}} \nsubseteq 0_1$ and $2_{\frac{1}{3}} \nsubseteq (0_1: X)$ since $(0_1: X)(2) = 0 \ngtr \frac{1}{3}$.

Remark 3.4:

Every Fzy small Two Absorbing module is a Fzy Soc-STA module module. However the converse is not true, for example:

Let $X: Z_{32} \rightarrow I$ such that $X(g) = \begin{cases} 1 \; if \; g \in Z_{32} \\ 0 \quad o.w \end{cases}$

It is clear X is a fzy module of Z_{32} as a Z-module.

$$0_1(g) = \begin{cases} 1 \; if \; g = 0 \\ 0 \; if \; g \neq 0 \end{cases}$$

$$Fzy - Soc(X)(g) = \begin{cases} \frac{1}{3} \; if \; g \in (\overline{16}) \\ 0 \; if \; g \notin (\overline{16}) \end{cases}$$

$$\big(0_1 + Fzy - Soc(X)\big)(g) = \begin{cases} 1 \; if \; g \in (16) \\ 0 \; if \; g \notin (16) \end{cases}$$

$$(0_1: X)(g) = \begin{cases} \frac{1}{6} \; if \; g \in 32Z \\ 0 \; if \; g \notin 32Z \end{cases}$$

X is Fzy Soc-STA module since $2_{\frac{1}{4}} \cdot 2_{\frac{1}{4}} \cdot \bar{8}_{\frac{1}{3}} = 0_{\frac{1}{4}} \subseteq 0_1$ and $< \bar{8}_{\frac{1}{3}} > \ll X$, then $2_{\frac{1}{4}} \cdot 8_{\frac{1}{3}} = 16_{\frac{1}{4}} \subseteq 0_1 + F - Soc(X)$, since $\big(0_1 + F - Soc(X)\big)(16) = 1 > \frac{1}{4}$, but X is not a Fzy small Two Absorbing module since $2_{\frac{1}{4}} \cdot 8_{\frac{1}{3}} = 16_{\frac{1}{4}} \nsubseteq 0_1$ and $2_{\frac{1}{4}} \cdot 2_{\frac{1}{4}} = 4_{\frac{1}{4}} \nsubseteq (0_1: X)$ since $(0_1: X)(16) = 0 \ngtr \frac{1}{4}$.

Proposition 3.5:

Let X be a fzy module of a T-module D, if X is Fzy Two Absorbing module, then X is Fzy Soc-STA module.
Proof:
Let $s_a y_b x_v \subseteq 0_1$ for fzy singletons s_a, y_b of T and $< x_v > \ll X$ where a, b, v ∈I, but X is Fzy Two Absorbing module, then $s_a x_v \subseteq 0_1$ or $y_b x_v \subseteq 0_1$ or $s_a y_b \subseteq (0_1 : X)$, hence $s_a x_v \subseteq 0_1 + fzy - Soc(X)$ or $y_b x_v \subseteq 0_1 + fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + fzy - Soc(X) : X)$, then 0_1 is Fzy Soc-STA sub-module. Thus, X is Fzy Soc-STA module.
Remark 3.6:
The converse of proposition 3.5, is not true in general, for example:
Let $X: Z_{12} \to I$ such that $X(g) = \begin{cases} 1 \; if \; g \in Z_{12} \\ 0 \quad\quad o.w \end{cases}$
It is clear X is fzy module of Z_{12} as Z-module.
Let $0_1(\text{g}) = \begin{cases} 1 \; if \; g = 0 \\ 0 \; if \; g \neq 0 \end{cases}$

$$(Fzy - Soc(X))(g) = \begin{cases} \frac{1}{3} \; if \; g \in (\bar{2}) \\ 0 \; if \; g \notin (\bar{2}) \end{cases}$$

$$(0_1 + fzy - Soc(X))(g) = \begin{cases} 1 \; if \; g \in (\bar{2}) \\ 0 \; if \; g \notin (\bar{2}) \end{cases}$$

$$(0_1 : X)(\text{g}) = \begin{cases} \frac{1}{8} \; if \; g \in 12Z \\ 0 \; if \; g \notin 12Z \end{cases}$$

Where $(0_1 : X) = 12Z$
X is Fzy Soc-STA module since $2_{\frac{1}{2}} \cdot 1_{\frac{1}{6}} \cdot 6_{\frac{1}{7}} = 0_{\frac{1}{7}} \subseteq 0_1, < 6_{\frac{1}{7}} > \ll X$ then $2_{\frac{1}{2}} \cdot 6_{\frac{1}{7}} = 0_{\frac{1}{7}} \subseteq 0_1 + F - Soc(X)$, $1_{\frac{1}{6}} \cdot 6_{\frac{1}{7}} = 6_{\frac{1}{7}} \subseteq 0_1 + F - Soc(X)$ and $2_{\frac{1}{2}} \cdot 1_{\frac{1}{6}} = 2_{\frac{1}{6}} \subseteq (0_1 + F - Soc(X) : X)$, but X is not Fzy Two Absorbing module since $2_{\frac{1}{2}} \cdot 3_{\frac{1}{6}} \cdot \bar{2}_{\frac{1}{7}} \subseteq 0_{\frac{1}{7}} \subseteq 0_1$, but $2_{\frac{1}{2}} \cdot \bar{2}_{\frac{1}{7}} = 4_{\frac{1}{7}} \nsubseteq 0_1$, $3_{\frac{1}{6}} \cdot \bar{2}_{\frac{1}{7}} = 6_{\frac{1}{7}} \nsubseteq 0_1$ and $2_{\frac{1}{2}} \cdot 3_{\frac{1}{6}} = 6_{\frac{1}{6}} \nsubseteq (0_1 : X)$.

Proposition 3.7:
Let X be a Fzy hollow module of a T-module D and $Fzy - Soc(X) \subseteq 0_1$, then X is a Fzy Two Absorbing module if and only if X is Fzy Soc-STA module.
Proof:
(⇒) By Proposition (3.5), we get result.
(⇐) Let X be a Fzy Soc-STA module. To prove X is a Fzy Two Absorbing module. Let $s_a y_b x_v \subseteq 0_1$ for fzy singletons s_a, y_b of T where a, b, v ∈I, since X is Fzy hollow, then $< x_v > \ll X$, and X is a Fzy Soc-STA module, so that $s_a x_v \subseteq 0_1 + Fzy - Soc(X)$ or $y_b x_v \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + Fzy - Soc(X) : X)$, then $s_a x_v \subseteq 0_1$ or $y_b x_v \subseteq 0_1$ or $s_a y_b \subseteq (0_1 : X)$ since $F - Soc(X) \subseteq 0_1$. Thus, X is Fzy Two Absorbing module.
Proposition 3.8:
Let X be a Fzy-chained module of a T-module D, then X is a Fzy Soc-STA module if and only if X is a Fzy Two Absorbing module.
Proof:
Since X is a Fzy chained module, then every proper fzy sub-modules P, K of X either $P \subseteq K$ or $K \subseteq P$, hence $P + K \neq X$, so that every fzy submodule is a Fzy small of X, thus X is a Fzy hollow module and by Proposition (3.6), we get the desired.

Proposition 3.9:
Let X be a Fzy faithful finitely generated cyclic module of a T-module D. If X is a Fzy Soc-STA module iff $(0_1 : X)$ is a Fzy Soc-STA ideal of T.
Proof:
⇒)Let $s_a y_b r_m \subseteq (0_1 : X)$ for F-singletons s_a, y_b, r_m of T and $< r_m >$ is Fzy small ideal of T where a, b, m ∈I , hence $s_a y_b r_m X \subseteq 0_1$, then $s_a y_b (r_m x_v) \subseteq 0_1$ so that, $< r_m x_v > \ll X$ for each $x_v \subseteq X$, v ∈I, since X is Fzy Soc-STA module then $s_a r_m x_v \subseteq 0_1 + Fzy - Soc(X)$ or $y_b r_m x_v \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + Fzy - Soc(X) : X)$. So that, $s_a r_m X \subseteq 0_1 + Fzy - Soc(X)$ or $y_b r_m X \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b X \subseteq 0_1 + Fzy - Soc(X)$ since X is cyclic Fzy module, then X is multiplication Fzy module by Proposition (2.17), then $0_1 = (0_1 : X)X$ and X is faithful multiplication Fzy module, hence by Lemma (2.18), we have $Fzy - Soc(X) = Fzy - Soc(T)X$. So that either $s_a r_m X \subseteq (0_1 : X)X + Fzy - Soc(T)X$ or $y_b r_m X \subseteq (0_1 : X)X + Fzy - Soc(T)X$ or $s_a y_b X \subseteq (0_1 : X)X + Fzy - Soc(T)X$, then by Proposition(2.22), we have

either $s_a r_m \subseteq (0_1 : X) + Fzy - Soc(T)$ or $y_b r_m \subseteq (0_1 : X) + Fzy - Soc(T)$ or $s_a y_b \subseteq (0_1 : X) + Fzy - Soc(T)$. Thus, $(0_1 : X)$ is a Fzy Soc-STA ideal of T .
$\Leftarrow$)let $s_a y_b x_v \subseteq 0_1$ for fzy singletons s_a, y_b of T and $< x_v > \ll X$ where a, b, v ∈I, since X is Fzy cyclic module, then $< x_v >= r_m h_n$, for fzy singleton r_m of T and $h_n \in X$ m, n ∈I, so that $s_a y_b r_m h_n \subseteq 0_1$, hence $s_a y_b r_m \subseteq (0_1 : h_n) \subseteq (0_1 : X)$. So that, $s_a y_b r_m \subseteq (0_1 : X)$, but $(0_1 : X)$ is Fzy Soc-STA ideal of T . hence, $s_a r_m \subseteq (0_1 : X) + Fzy - Soc(T) \subseteq (0_1 + Fzy - Soc(X) : X)$ or $y_b r_m \subseteq (0_1 : X) + Fzy - Soc(T) \subseteq (0_1 + Fzy - Soc(X) : X)$ or $s_a y_b \subseteq (0_1 : X) + Fzy - Soc(T) \subseteq (0_1 + Fzy - Soc(X) : X)$, then $s_a r_m h_n \subseteq 0_1 + Fzy - Soc(X)$ or $y_b r_m h_n \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + Fzy - Soc(X) : X)$, hence $s_a x_v \subseteq 0_1 + Fzy - Soc(X)$ or $y_b x_v \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + Fzy - Soc(X) : X)$. Thus, X is Fzy Soc-STA module.

Proposition 3.10:
Let X be a fzy module of a T-module D, then X is a Fzy Soc-STA module if and only if $(0_1 : s_a x_v) \subseteq (0_1 + Fzy - Soc(X) : x_v)$ or $(0_1 : s_a x_v) \subseteq (0_1 + Fzy - Soc(X) : s_a X)$ for each fzy singleton s_a of T , $s_a x_v \nsubseteq 0_1 + Fzy - Soc(X), < x_v > \ll X$ where a, v ∈I.
Proof:
$\Rightarrow$) Let $y_b \subseteq (0_1 : s_a x_v)$ for fzy singleton y_b of T , b ∈I, hence $s_a y_b x_v \subseteq 0_1$, $< x_v > \ll X$ and $s_a x_v \nsubseteq 0_1 + Fzy - Soc(X)$, so that $y_b x_v \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + Fzy - Soc(X) : X)$ since X is a Fzy Soc-STA module, therefore, $y_b \subseteq (0_1 + Fzy - Soc(X) : x_v)$ or $y_b \subseteq (0_1 + Fzy - Soc(X) : s_a x_v)$. Thus $(0_1 : s_a x_v) \subseteq (0_1 + Fzy - Soc(X) : x_v)$ or $(0_1 : s_a x_v) \subseteq (0_1 + Fzy - Soc(X) : s_a X)$
$\Leftarrow$) To prove that X is a Fzy Soc-STA module. Let $s_a y_b x_v \subseteq 0_1$ and $< x_v > \ll X$ for fzy singletons s_a, y_b of T where a, b, v ∈I. Suppose that, $s_a x_v \nsubseteq 0_1 + Fzy - Soc(X)$, then $y_b \subseteq (0_1 : s_a x_v)$ so that $y_b \subseteq (0_1 + Fzy - Soc(X) : x_v)$ or $y_b \subseteq (0_1 + Fzy - Soc(X) : s_a X)$ by hypothesis, hence $y_b x_v \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b X \subseteq 0_1 + Fzy - Soc(X)$; that is $y_b x_v \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + Fzy - Soc(X) : X)$. Thus, X is a Fzy Soc-STA module.

Proposition 3.11:
Let X be a fzy module of a T-module D and $Fzy - Soc(X)$ is Fzy Soc-STA sub-module of X. Then X is a Fzy Soc-STA module.
Proof:
Let $s_a y_b x_v \subseteq 0_1$ for fzy singletons s_a, y_b of T and $< x_v > \ll X$ where a, b, v ∈I, then $s_a y_b x_v \subseteq 0_1 + Fzy - Soc(X)$, but $0_1 + Fzy - Soc(X)$ is Fzy Soc-STA sub-module. So that either $s_a x_v \subseteq 0_1 + Fzy - Soc(X) + Fzy - Soc(X) = 0_1 + Fzy - Soc(X)$ or $y_b x_v \subseteq 0_1 + Fzy - Soc(X)$ or $s_a y_b \subseteq (0_1 + Fzy - Soc(X) : X)$. Thus, X is Fzy Soc-STA module.

Conflicts Of Interest

There are no conflicts of interest.

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