Babylonian Journal of Mathematics
Vol. **2025**, **pp**. 12–17
DOI:https://doi.org/10.58496/BJM/2025/002; ISSN:3006-113X
https://mesopotamian.press/journals/index.php/mathematics

Research Article

An Enhanced Strategy for Computing Minimum Path and Cut Sets to Improve Complex Network Reliability

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ARTICLE INFO

Article History

Received 18 Dec 2024
Revised: 02 Jan 2025
Accepted 24 Jan 2025
Published 25 Feb 2025

Keywords
Reliability
Minimum paths
Minimum cut sets
Complex systems

ABSTRACT

Improving the reliability of complex systems is an urgent requirement due to their wide applications in various sciences. This study presents the development of two methods to compute minimum paths and minimum cut sets analytically using the connectivity matrix. The proposed method is characterized by its simplified procedures free of complex computational procedures that may be subject to errors, in addition to the speed of completion. The accuracy of calculating these paths ensures that the reliability value of the system components can be extracted and thus facilitates the optimization process with a reliable result. Moreover, the proposed method was applied to two types of complex systems and the results were very accurate. Finally, this method opens up prospects for improving the reliability of highly complex systems more smoothly and with promising results.

1. INTRODUCTION

Finding the Minimum Path and Cut Sets is essential to improving the reliability of complex systems. Accordingly, the continuity of system operation and control can be ensured by identifying the smallest component combinations or critical paths of the components of those systems. On the other hand, complex systems often contain many intricately interconnected components, and thus identifying those component combinations reduces the failure of those systems by restructuring and redesigning those systems to operate without interruption and with high efficiency[1], [2]. In addition, the development of reliability-finding methods is mainly based on the use of those component combinations, as in the branch and boundary algorithm, where this algorithm is used to analyze the reliability of the system by analyzing the reliability of each component and thus the nature of its impact on the reliability of the system as a whole. Therefore, improving the reliability of systems depends primarily on the use of these combinations, through which the priority of each component is determined based on its impact on the reliability of the system, which provides a comprehensive view for designers of complex systems to determine the component most in need of maintenance, thus improving the reliability of the system and developing more efficient methods for reliability analysis [3]-[7]. In this research, two methods were developed to find the minimum cutting paths based on the contact matrix and applied to two types of complex systems. The results were very promising, which contributes to the development and improvement of complex systems and thus increasing their reliability.

2. BASIC CONCEPTS

This section covers the basic concepts that are the cornerstone for finding minimum paths and minimum cut sets for complex systems:

Definition 2.1

A path (P) is a collection of units that, while in operation, ensure that the system works by connecting the start and end nodes via working edges [8].

Definition 2.2

A path's number of edges (or units) is called a path length (P L) [9].

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Definition 2.3
Let G be a graph that has n-node. Then, a path that has no edges that can be deleted without cutting the connection between the start and end nodes so that P L < n is called a minimum path (MP) [9]- [11].
Definition 2.4
Let C be a collection of edges in graph G. If all of C edges are removed from G, then G is said to be disconnected, and C is a cut set of G [8]- [11].
Definition 2.5
The quantity of edges in a cut set is referred to as the order of the cut set, abbreviated as OC [11].
Definition 2.6
A minimum cut set (MC) of a graph with n nodes is a cut set that does not contain any other cut set such that OC < n, [7]- [12].
Definition 2.7
An adjacency matrix is added to the identity matrix Im to create a connection matrix (CM), which is expressed as CM = A + Im, which substitutes the zeros in diagonal of an adjacency matrix with ones [13]-[19].

3. PROPOSED STRATEGY

This section summarizes the strategy used to find minimum paths and minimum cut sets for complex systems by the following steps:

3.1 Minimum Paths Enumeration

- Step 1. Construct the connection matrix.
- Step 2. Add connection matrix to identity matrix ($C = CM + I_n$).
- Step 3. Remove first column and last row of C to construct $n - 1 \times n - 1$ matrix say A.
- Step 4. Expand the determinant of A. Each term of the determinant represents a minimum path [9]-[13].

3.2 Minimum Cut Sets Enumeration

- Step 1. Construct the connection matrix.
- Step 2. Collect all the elements in the first row- a source minimum cut, and in the last
- column-a destination minimum cut.

Step 3. Form a set S of all column's combinations of order 1 to $(n - 3)$, with columns 2 to $(n - 1)$.
Delete a combination by observing:
a. If the combination consists of only those column having zeros in the first row,
b. If the combination consists of those rows having nonzero entries in the last column.

- Step 4. Take one combination; collect all the links-labels appearing in the row(s) corresponding to row#1 + this combination without considering columns represented by this combination. This combination will provide another cut set.
- Step 5. Repeat step 4 for all remaining combinations [9]-[13].

4. IMPLEMENTATION

In this section, the proposed methods will be implemented on two types of complex systems, as shown in Fig. 1 and Fig. 2.

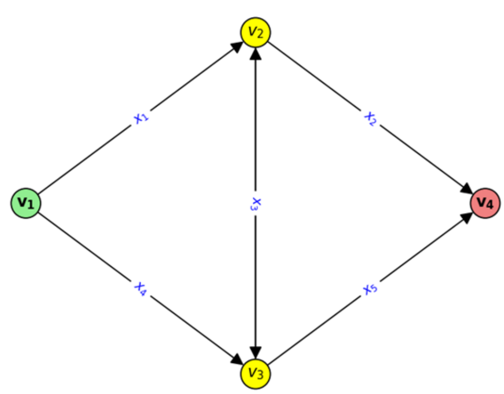

Fig 1. Complex system with five components

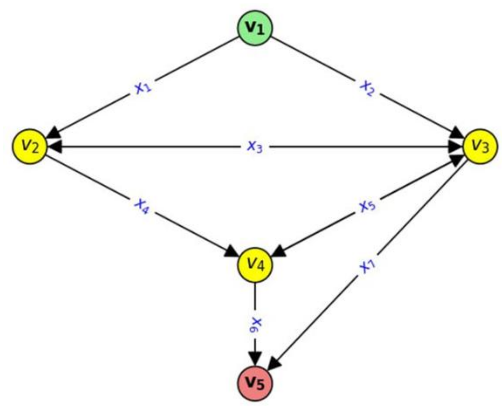

Fig 2. Complex system with seven components

4.1 Finding minimum paths

For the system shown in Fig. 1, the steps for finding the minimum paths are as follows:
Step 1. Construct the connection matrix.

$$CM = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} 0 & x_1 & x_4 & 0 \\ 0 & 0 & x_3 & x_2 \\ 0 & x_3 & 0 & x_5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2. Add connection matrix to identity matrix ($C = CM + I_n$).

$$CM + I_4 = \begin{bmatrix} 0 & x_1 & x_4 & 0 \\ 0 & 0 & x_3 & x_2 \\ 0 & x_3 & 0 & x_5 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_4 & 0 \\ 0 & 1 & x_3 & x_2 \\ 0 & x_3 & 1 & x_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3. Remove first column and last row of C to construct $n - 1 \times n - 1$ matrix say A.

$$A = \begin{bmatrix} x_1 & x_4 & 0 \\ 1 & x_3 & x_2 \\ x_3 & 1 & x_5 \end{bmatrix}$$

Step 4. Expand the determinant of A. Each term of the determinant represents a minimum path.

$$|A| = \begin{vmatrix} x_1 & x_4 & 0 \\ 1 & x_3 & x_2 \\ x_3 & 1 & x_5 \end{vmatrix} = x_1 \begin{vmatrix} x_3 & x_2 \\ 1 & x_5 \end{vmatrix} - x_4 \begin{vmatrix} 1 & x_2 \\ x_3 & x_5 \end{vmatrix}$$
$$= x_1(x_3x_5 - x_2) - x_4(x_5 - x_3x_2)$$
$$= x_1x_3x_5 - x_1x_2 - x_4x_5 + x_4x_3x_2$$

So, the minimum paths are:

$$MP_1 = x_1x_2\ ,\ MP_2 = x_4x_5,\ MP_3 = x_1x_3x_5\ ,\ MP_4 = x_2x_3x_4$$

As for the system shown in Fig 2, the steps for finding the minimum paths are as follows:

$$\text{Step 1:} \quad CM = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & 0 \\ 0 & x_3 & 0 & x_5 & x_7 \\ 0 & 0 & x_5 & 0 & x_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\text{Step 2:} \quad CM + I_5 = \begin{bmatrix} 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & 0 \\ 0 & x_3 & 0 & x_5 & x_7 \\ 0 & 0 & x_5 & 0 & x_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix}1 & x_1 & x_2 & 0 & 0\\ 0 & 1 & x_3 & x_4 & 0\\ 0 & x_3 & 1 & x_5 & x_7\\ 0 & 0 & x_5 & 1 & x_6\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Step 3: $$A=\begin{bmatrix}x_1 & x_2 & 0 & 0\\ 1 & x_3 & x_4 & 0\\ x_3 & 1 & x_5 & x_7\\ 0 & x_5 & 1 & x_6\end{bmatrix}$$

Step 4: $$|A|=\begin{vmatrix}x_1 & x_2 & 0 & 0\\ 1 & x_3 & x_4 & 0\\ x_3 & 1 & x_5 & x_7\\ 0 & x_5 & 1 & x_6\end{vmatrix}$$

$$|A|=x_1\begin{vmatrix}x_3 & x_4 & 0\\ 1 & x_5 & x_7\\ x_5 & 1 & x_6\end{vmatrix}-x_2\begin{vmatrix}1 & x_4 & 0\\ x_3 & x_5 & x_7\\ 0 & 1 & x_6\end{vmatrix}$$

$$|A|=x_1\begin{vmatrix}x_3 & x_4 & 0\\ 1 & x_5 & x_7\\ x_5 & 1 & x_6\end{vmatrix}\begin{matrix}x_3 & x_4\\ 1 & x_5\\ x_5 & 1\end{matrix}-x_2\begin{vmatrix}1 & x_4 & 0\\ x_3 & x_5 & x_7\\ 0 & 1 & x_6\end{vmatrix}\begin{matrix}1 & x_4\\ x_3 & x_5\\ 0 & 1\end{matrix}$$

$$|A|=x_1x_3x_5x_6+x_1x_4x_5x_7-x_1x_3x_7-x_1x_4x_6-x_2x_5x_6+x_2x_7+x_2x_3x_4x_6$$

So, the minimum paths are:

$MP_1=x_2x_7$, $MP_2=x_1x_3x_7$, $MP_3=x_1x_4x_6$, $MP_4=x_2x_5x_6$, $MP_5=x_1x_3x_5x_6$, $MP_6=x_1x_4x_5x_7$, $MP_7=x_2x_3x_4x_6$

4.2. Finding minimum cut sets

For the system shown in Fig. 1, the steps for finding the minimum cut sets are as follows:

Step 1: Construct the connection matrix:

$$CM=\begin{bmatrix}0 & x_1 & x_4 & 0\\ 0 & 0 & x_3 & x_2\\ 0 & x_3 & 0 & x_5\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Step 2: The elements in the first row $MC_1=\{x_1x_4\}$

The elements in the last column $MC_2=\{x_2x_5\}$

Step 3. Form a set S of all column's combinations of order 1 to $(4-3=1)$, with columns 2 to $(4-1=3)$.

Order 1 only Columns 2 to 3 ⇨ 2,3

$S=\{\{2\},\{3\}\}$

$\{2\}$: column 2 has $x_1\neq 0$ in the first row. Remaining in S.

$\{3\}$: column 3 has $x_2\neq 0$ in the first row. Remaining in S.

This means that S has not changed.

Step 4: $\{2\}$: $CM_{1,2}=\begin{bmatrix}0 & x_1 & x_4 & 0\\ 0 & 0 & x_3 & x_2\end{bmatrix}$

The elements of $CM_{1,3}$ are x_1, x_3, x_4, x_5. Delete x_4 which is an element of columns 1 and 3 to get: $MC_4=\{x_1x_3x_5\}$.

So, the minimum cut sets are:

$MC_1=\{x_1x_4\}$

$MC_2=\{x_2x_5\}$

$MC_3=\{x_2x_3x_4\}$

$MC_4=\{x_1x_3x_5\}$

As for the system shown in Fig 2, the steps for finding the minimum cut sets are as follows:

Step 1: $$CM=\begin{bmatrix}0 & x_1 & x_2 & 0 & 0\\ 0 & 0 & x_3 & x_4 & 0\\ 0 & x_3 & 0 & x_5 & x_7\\ 0 & 0 & x_5 & 0 & x_6\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$MC_1=\{x_1x_2\}$ and $MC_2=\{x_6x_7\}$

Step 2: Here $n = 5 \Rightarrow$ Order 1 to $5 - 3 = 2$ Columns 2 to $(n-1) \Rightarrow$ 2,3,4

$$S = \{\{2\},\{3\},\{4\},\{2,3\},\{2,4\},\{3,4\}\}$$

$\{2\}$: column 2 has $x_1 \neq 0$ in the first row. Remaining in S.
$\{3\}$: column 3 has $x_2 \neq 0$ in the first row. Remaining in S.
$\{4\}$: column 4 has zero in the first row. Delete it from S.
$\{2,3\}$: rows 2 and 3 having zero entries in the last column. Remaining in S.
$\{2,4\}$: rows 2 and 4 having zero entries in the last column. Remaining in S.
$\{3,4\}$: rows 3 and 4 having nonzero entries in the last column. Delete it from S.
Update S
$S = \{\{2\},\{3\},\{2,3\},\{2,4\},\}$

$$\{2\}:\ CM_{1,2} = \begin{bmatrix} 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & 0 \end{bmatrix}$$

The elements of $CM_{1,2}$ are x_1, x_2, x_3, x_4. Delete x_1 which is an element of columns 1 and 2 to get: $MC_3 = \{x_2x_3x_4\}$.

$$\{3\}:\ CM_{1,3} = \begin{bmatrix} 0 & x_1 & x_2 & 0 & 0 \\ 0 & x_3 & 0 & x_5 & x_7 \end{bmatrix}$$

The elements of $CM_{1,3}$ are x_1, x_2, x_3, x_5, x_7. Delete x_2 which is an element of columns 1 and 3 to get: $MC_4 = \{x_1x_3x_5x_7\}$.

$$\{2,3\}:\ CM_{1,2,3} = \begin{bmatrix} 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & 0 \\ 0 & x_3 & 0 & x_5 & x_7 \end{bmatrix}$$

The elements of $CM_{1,2,3}$ are $x_1, x_2, x_3, x_4, x_5, x_7$. Delete x_1, x_2, x_3 which are elements of columns 1, 2 and 3 to get: $MC_5 = \{x_4x_5x_7\}$.

$$\{2,4\}:\ CM_{1,2,4} = \begin{bmatrix} 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & x_4 & 0 \\ 0 & 0 & x_5 & 0 & x_6 \end{bmatrix}$$

The elements of $CM_{1,2,4}$ are $x_1, x_2, x_3, x_4, x_5, x_6$. Delete x_1, x_4 which are elements of columns 1, 2 and 4 to get: $MC_6 = \{x_2x_3x_5x_6\}$.
So, the minimum cut sets are:
$MC_1 = \{x_1x_2\}$, $MC_2 = \{x_6x_7\}$, $MC_3 = \{x_2x_3x_4\}$, $MC_4 = \{x_1x_3x_5x_7\}$, $MC_5 = \{x_4x_5x_7\}$, $MC_6 = \{x_2x_3x_5x_6\}$

5. CONCLUIONS

The proposed method has given promising results in a simplified and computationally cost-free manner as well as in its accuracy. The method's accuracy in calculating minimum paths and minimum cut sets allows for finding the reliability of system components and the overall system with high efficiency and thus discovering critical components and the need for system improvement. The results were very accurate when applied to two types of complex systems. Finally, finding minimum paths and minimum cut sets allows improving the reliability of very complex systems through several options including upgrading the systems according to their design or identifying critical components that require maintenance or improvement.

Conflicts Of Interest

The authors declare no conflicts of interest.

Funding

The paper has not received any funding.

Acknowledgment

The authors express their gratitude and appreciation to the team of Babylonian Journal for their cooperation in publishing this paper.

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