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Research Article

Existence and Uniqueness Theorem of Multi-Dimensional Integro-Differential Equations With Fractional Differointegrations

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ABSTRACT

The aim of this paper is to establish the existence and uniqueness theorems of multi-dimensional conformable fractional partial integro-differential equations theorem ,The Picard iterative sequence converges when the provided functions are assumed to be sufficiently smooth. This is accomplished by demonstrating that the sequence is a Cauchy sequence in the entire normed space of continuous functions, implying the presence of a solution.
for generalized contraction mappings with respect to w-distances in complete metric spaces.

1. INTRODUCTION

Fractional calculus was introduced by mathematicians like Leibniz and Euler in the 18th century, but it was not until the 20th century that it began to be studied more systematically. In the 1970s, researchers like Samko, Kilbas, and Marichev developed a theory of fractional integrals and derivatives that was based on the Riemann-Liouville definition of fractional derivatives. Fractional calculus was introduced by mathematicians like Leibniz and Euler in the 18th century, but it was not until the 20th century that it began to be studied more systematically. In the 1970s, researchers like Samko, Kilbas, and Marichev developed a theory of fractional integrals and derivatives that was based on the Riemann-Liouville definition of fractional derivatives [4,5].

In recent years, there has been growing interest in extending fractional calculus to higher dimensions, and this has led to the development of the theory of CFPIDEs. These equations are a generalization of fractional partial differential equations, and they involve both fractional derivatives and fractional integrals. The first Existence and Uniqueness Theorem for CFPIDEs was established in 2019 by Jafari et al. In their work, they proved that under certain conditions, CFPIDEs have a unique solution that can be expressed in terms of a series solution. The conditions they imposed were based on a new concept of conformable fractional derivative, which is a modification of the Riemann-Liouville fractional derivative that is better suited for higher-dimensional problems. Since the publication of Jafari et al.'s result, there has been a growing interest in the study of CFPIDEs and their applications in fields like physics, engineering, and finance. Researchers have continued to refine the theory and develop new techniques for solving CFPIDEs, and it is likely that this area of mathematics will continue to be an active area of research for years to come [6,7,8].

Various phenomena's of viscoelastic its, diffusion procedures, relaxation vibrations, electrochemistry, etc. are successfully described by fractional differential equations (FDEs) and therefor The researchers tried to suggest several types of fractional operators to describe more accurately these phenomena's since fractional order deferential equations are generalization of integer order deferential equations to non-integer order ones. The fractional calculus was bounded up with fractional integrals

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obtained by iterating an integral to get the n -th order integral and after that replacing n by any number, and then by using the classical method the corresponding derivatives were defined, proposed a new derivative with real orders, and hence different definitions of fractional integrals and derivatives are proposed and In this work the conformable fractional calculus will be considered. Which is due to its well-behaved properties and the close relationship with first order derivative, conformable derivatives and integrals and has exerted a tremendous fascination on researchers [1, 2].

The most well-known fixed-point results in the metrical fixed point theory are on based Banach's contraction mapping principle, Moreover, this principle has many applications not only of the various branches in mathematical topics, but also in economics, chemistry, biology, computer science, engineering, and others. Based on the mentioned impact, it was developed extensively by several researchers [3]. This enables a researcher to choose the most suitable operator in order to describe the dynamics in a real world problem.

2. THEORETICAL BACKGROUND FOR BANACH FIXED POINT THEOREM

This section introduces some basic concepts that are necessary for establishing the theorem on the existence and uniqueness for the solution and to find sufficient conditions that satisfies the Lipschitz condition. The main aim of this section is to provide some necessary definitions and a theorem including Banach fixed point theorem, which are used throughout this thesis. Now, we will start with the basic concepts related to this work, in which more elementary concepts of under graduate study will be omitted.

Definition 2.1. [32] Let $(X, \|\cdot\|)$ be a normed space and $T: X \to X$ be a mapping. A point $x \in X$ for which $Tx = x$ is called a fixed point of T.

Theorem 2.2. [9]Let $(X, \|\cdot\|)$ be a complete normed space and let the mapping $: X \to X$ be a contraction mapping, then T has exactly one fixed point.

An additional definition which is also necessary in the proof the existence and uniqueness of solution of the considered IDE in this work is the next definition of Lipschitzian function..

Remark 2.5. The space $C_t^{m-1}([a,b] \times [0,T])$ will be used to denote the Banach space of all continuous real valued functions u defined on $[a,b] \times [0,T]$ with continuous m^{th} order partial derivatives with respect to t.

3. EXISTENCE AND UNIQUENESS OF MULTI-DIMENSIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVE

One of the most important tasks in this thesis is to find the approximate solution of multi-dimensional integro-differential equations to satisfy the existence and uniqueness theorem. As we said above, we shall use the fixed point principle based on Banach fixed point theorem or the contraction mapping. which has the form:

Consider the generalized two-dimensional linear integro-differential equation of fractional order:

$$T_x^{\alpha} u(x,y) = g(x,y) + I_s^{\beta} I_t^{\gamma} [K(x,y,s,t) u(s,t)]$$

where g and K are enough smooth functions, $x \in [a,b], y \in [c,d]$, T_x^{α} is the conformable fractional derivative of order $\alpha \in \mathbb{R}^+$with respect to x, and $I_s^{\beta}, I_t^{\gamma}$ are the conformable fractional integrals of orders $\beta, \gamma \in (0,1]$ with respect to s and t, respectively.

The existence and uniqueness of the solution of Eq. (1) will be proved based on Picard successive iterative approach.

Theorem (1):

Suppose that

$$\Omega = \{(x,y) \mid x \geq a, y \geq c\}$$

and let g and K be enough smooth functions on Ω.

Then Eq. (1) has a solution.

Proof:

In order to be able to apply the Picard iterative method on Eq. (1), we must first drop the derivative from the left-hand side of Eq. (1).

For this purpose, apply the conformable fractional integral with respect to x of order α to both sides, getting:

$$I_x^{\alpha} T_x^{\alpha} u(x,y) = I_x^{\alpha} g(x,y) + I_x^{\alpha} I_s^{\beta} I_t^{\gamma} [K(x,y,s,t) u(s,t)]$$

and hence, from the properties of conformable fractional order differentiation and integration, we get:

$$u(x,y) = u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y) dt + \int_a^x \int_a^w \int_c^y (s-a)^{\beta-1} (t-c)^{\gamma-1} [K(w,y,s,t) u(s,t)] dt ds dw \ldots (2)$$

In order to prove the existence of the solution of Eq. (1), which is equivalent to proving the existence of a solution of Eq. (2), we use the Picard iteration method on Eq. (2), given by ...

From Eq. (2), we define the iterative sequence as

$$u_{n+1}(x,y) = u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y)dt + \int_a^x \int_a^w \int_c^y (w-a)^{\alpha-1} (s-a)^{\beta-1} (t-c)^{\gamma-1} [K(w,y,s,t)u_n(s,t)]dtdsdw \ldots (3)$$

where $u_0(x,y)$ is the initial approximate solution, which is taken for simplicity to begin as some assumed function often equal to zero or different from the initial solution of Eq. (1).
Now, the principle of mathematical induction will be used to prove that

$$|u_{n+1}(x,y) - u_n(x,y)| \leq$$

If $n = 0$, and since g and K are smooth functions, then $|g| \leq G_0, |K| \leq K_0$, where G_0 and K_0 are positive constants. Hence, when starting with $u_0(x,y) = 0$, we get from Eq. (3):

$$u_1(x,y) = u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y)dt \text{ (approximate solution)}$$

Therefore,

$$|u_1(x,y) - u_0(x,y)| = \left| \int_a^x (t-a)^{\alpha-1} g(t,y)dt \right|$$

and hence,

$$|u_1(x,y) - u_0(x,y)| \leq \int_a^x |t-a|^{\alpha-1} |g(t,y)|dt$$
$$\leq G_0 \int_a^x (t-a)^{\alpha-1} dt = G_0 \frac{(x-a)^\alpha}{\alpha}$$

If $n \geq 1$, then:

$$|u_{n+2}(x,y) - u_{n+1}(x,y)| = | u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y)dt + \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1} K(w,y,s,t)u_n$$
$$\leq \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1} |K(w,y,s,t)||u_{n+1}(s,t) - u_n(s,t)|dtdsdw$$
$$\leq \frac{K_0 G_0}{\alpha} \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta+\alpha-1}(t-c)^{\gamma-1} dtdsdw$$
$$= \frac{K_0 G_0 (y-c)^\gamma}{\alpha\gamma} \int_a^x (w-a)^{\beta+\alpha-1} dw$$
$$= \frac{K_0 G_0 (y-c)^\gamma (x-a)^{\beta+\alpha}}{\alpha\gamma(\beta+\alpha)(\beta+2\alpha)}$$

If $n = 2$, then:

$$|u_3(x,y) - u_2(x,y)| = | u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y)dt + \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1} K(w,y,s,t)u_2(s,t)$$
$$\leq \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1} |K(w,y,s,t)||u_2(s,t) - u_1(s,t)|dtdsdw$$

$$\leq \frac{K_0^2 G_0}{\alpha} \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1}|u_1(s,t)|dtdsdw$$
$$\leq \frac{K_0^2 G_0}{\alpha} \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1}(t-c)^{\gamma}dtdsdw$$
$$= \frac{K_0^2 G_0 (y-c)^{2\gamma}}{\alpha 2\beta^2} \int_a^x \int_a^w \frac{(w-a)^{\beta+2\alpha-1}}{(\beta+\alpha)(\beta+2\alpha)}(s-a)^{\beta-1}dsdw$$
$$= \frac{K_0^2 G_0 (y-c)^{2\gamma}(x-a)^{2(\beta+\alpha)}}{\alpha 2\beta^2(\beta+\alpha)(\beta+2\alpha)}$$

Similarly, if $n = 3$, then we will have

$$|u_4(x,y) - u_3(x,y)| \leq \frac{K_0^3 G_0 (y-c)^{3\gamma}(x-a)^{3(\alpha+\beta)}}{3!\,\alpha^3\beta^3(\beta+\alpha)(\beta+2\alpha)(\beta+3\alpha)(4\beta+3\alpha)}$$

Therefore, as n increases, we will have that $\{u_n\}$ is a Cauchy sequence in the space $C(\Omega)$, which is a complete normed space.

Thus, there exists $u(x,y)$ such that

$$u(x,y) = \lim_{n\to\infty} u_n(x,y), (x,y) \in \Omega$$

Therefore, when taking the limit as $n \to \infty$ of the iterative equation (3), we get ...

$$\lim_{n\to\infty} u_{n+1}(x,y) = u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y)dt + \iiint_{a,a,c}^{x,w,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t - c)^{\gamma-1}K(w,y,s,t)u(s,t)dtdsdw$$

and hence,

$$u(x,y) = u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y)dt + \iiint_{a,a,c}^{x,w,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1}K(w,y,s,t)u(s,t)dtdsdw$$

Therefore, $u(x,y)$ is a solution of Eq. (1).

Theorem (2):
Suppose that g and K in Eq. (1) are smooth functions, such that

$$|g| \leq G_0, |K| \leq K_0 < 1, K_0 G_0 \in \mathbb{R}^+.$$

Then the solution of Eq. (1) is unique.

Proof:
Suppose that $u^*(x,y) \in C(\Omega)$ is any other solution of Eq. (1), or equivalently Eq. (2). Hence,

$$u^*(x,y) = u_0(x,y) + \int_a^x (t-a)^{\alpha-1} g(t,y)dt + \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t - c)^{\gamma-1}K(w,y,s,t)u^*(s,t)dtdsdw \ldots (4)$$

Thus, subtracting Eq. (4) from Eq. (3), we get:

$$u_{n+1}(x,y) - u^*(x,y) = \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1}K(w,y,s,t)[u_n(s,t) - u^*(s,t)]dtdsdw$$

Hence, when $n \geq 0$, we get

$$|u_{n+1}(x,y) - u^*(x,y)| \leq K_0 \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1}|u_n(s,t) - u^*(s,t)|dtdsdw$$

If the initial approximate solution is $u_0(x,y) = 0$, then from the last inequality we have

$$|u_1(x,y) - u^*(x,y)| \le K_0 \sup_{(x,y)\in\Omega} |u^*(x,y)| \frac{(y-c)^\gamma (x-a)^{\beta+\alpha}}{\gamma\beta(\beta+\alpha)}$$

Also, if $n \ge 1$, it will imply that

$$|u_{n+1}(x,y) - u^*(x,y)| \le K_0 \sup_{(x,y)\in\Omega} \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1}|u_n(s,t) - u^*(s,t)|dtdsdw$$

$$\le K_0^2 \sup_{(x,y)\in\Omega} |u^*(x,y)| \iiint_{a,a,c}^{x,y,y} (w-a)^{\alpha-1}(s-a)^{\beta-1}(t-c)^{\gamma-1} \frac{(t-c)^\gamma (s-a)^\beta (w-a)^\alpha}{\gamma\beta(\beta+\alpha)} dtdsdw$$

$$= K_0^2 \sup_{(x,y)\in\Omega} |u^*(x,y)| \frac{(y-c)^{2\gamma+1}(x-a)^{2\beta+2\alpha+1}}{\gamma(2\gamma+1)\beta((\beta+\alpha)(2\beta+\alpha)(2\beta+2\alpha+1))}$$

and so, as $n \to \infty$, and since $|K_0| < 1$, with an increased denominator, we will have

$$|u_{n+1}(x,y) - u^*(x,y)| \to 0 \text{ as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} u_n(x,y) = u^*(x,y).$$

Since $\lim_{n\to\infty} u_n(x,y) = u(x,y)$, it follows that $u(x,y) = u^*(x,y)$.
Hence, the fractional conformable integro-differential equation (1) has a unique solution.

Conflicts Of Interest

The author's paper emphasizes that there are no conflicts of interest, either perceived or actual, that could impact the research integrity.

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