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Research Article

Graph-Theoretic Characterizations of Quasi-Idempotents in Full Order-Preserving Transformation Semigroup

Eze C.[1,], A.T Imam [1,], M.Balarabe [1,], Olaiya O.O [2, *,]

[1] *primary affiliation Ahmadu Bello University – Mathematics Pensioner quaters Hanyar Danyaro. Zaria, Zaria Kaduna, 810106, Nigeria*
[2] *Department of Mathematics,National Mathematical Centre,Sheda Kwali,Abuja, Nigeria.*

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ABSTRACT

This paper presents a digraph-theoretic extension of the characterization of quasi-idempotent in the semigroup On of full order-preserving transformations on a finite chain. Building on earlier results . that describe quasi-idempotent as those transformations α ∈ On satisfying α≠α^2=α^4, we provide a novel interpretation using the functional digraphs of such maps. We show that a transformation is quasi-idempotent if and only if each vertex in its associated digraph is either fixed or maps directly into a fixed point, and every non-trivial strongly connected component forms a 2-cycle. Furthermore, we prove that no directed path of the form v1 → v2 → v3 exists where all vertices are non-stationary. These findings offer a new perspective on the structure of On, bridging algebraic properties with graphical structure, and set the stage for visual and computational analysis of quasi-idempotent generation in transformation semigroups.

1. INTRODUCTION

Transformation semigroups play a foundational role in semigroup theory, much like permutation groups do in group theory. The full transformation semigroup Tn on a finite set Xn = {1, 2, . . ., n} provides a universal framework within which many semigroups can be embedded. A notable subsemigroup of Tn is the set On of all full order-preserving transformations, which has been studied extensively due to its regular structure and combinatorial richness, as outlined by Gomes and Howie [9]. Since Howie’s classical result [10] showing that every singular transformation in Tn \ Sn can be expressed as a product of idempotents, various investigations have focused on the ranks, generators, and structural properties of idempotent-generated semigroups within Tn [5, 11–14, 16, 19]. However, in certain semigroups such as inverse semigroups, the set of idempotents forms a subsemigroup and thus cannot serve as a generating set [15]. This led to the introduction of quasi-idempotent—elements α such that α≠α^2=α^4—which generalize the concept of idempotents while preserving useful generating properties. Garba and Imam [7] utilized quasi-idempotent to generate the symmetric inverse semigroup. Subsequent works demonstrated their effectiveness in other transformation structures: Madu and Garba [18] showed their role in generating order-preserving partial injections, while Bugay [3, 4] analyzed their ranks in ideals and partial transformation semigroups. Combinatorial properties of quasi-idempotent have also been explored in recent studies [17]. In a recent contribution, Imam et al. [2] characterized quasi-idempotent elements in On and proved that On is generated by quasi-idempotent of height n − 1. They further determined that such elements map each non-stationary block into a stationary block and established an upper bound on the minimal size of quasi-idempotent generating sets. In this paper, we extend this algebraic analysis by adopting a digraph-theoretic perspective. By associating each transformation with its functional digraph, we characterize quasi-idempotent in On using graph-theoretic conditions such as the absence of directed 3-paths among non-stationary vertices and the decomposition into stationary vertices and disjoint 2- cycles. This approach provides a structural viewpoint that complements the algebraic characterizations and supports further computational applications.

2. PRELIMINARIES

Definition 2.1. [15] Let Xn = {1, 2, . . . , n}. A full transformation on Xn is a function α : Xn → Xn. The set of all such transformations forms the full transformation semigroup Tn under composition.

Definition 2.2. [15] A transformation α ∈ Tn is said to be order-preserving if for all x, y ∈ Xn, whenever x ≤ y, it holds that α(x) ≤ α(y). The set of all order-preserving transformations in Tn is denoted by On.

Corresponding author. Email: ezewisdom8@gmail.com

Definition 2.3. [15] An element α ∈ On is called idempotent if α2 = α. It is called a quasi-idempotent if α≠α^2=α^4.

Definition 2.4. [21] The digraph of a transformation α ∈ Tn is the functional directed graph Γ = (V, E) where V = Xn and E = {(x, α(x)) | x ∈ Xn}.

3. MAIN RESULTS

Definition 3.1. A vertex v ∈ V in a functional digraph is called stationary if α(v) = v; otherwise, it is called non-stationary.
Definition 3.2. A 2-cycle is a pair of distinct vertices v, w ∈ V such that α(v) = w and α(w) = v.
Theorem 3.3. Let Γ = (V, E) be the functional digraph of an order-preserving trans- formation α ∈ On, where:

V = {1, 2, . . ., n},

E = {(x, α(x)) | x ∈ V}.

Then the following are equivalent:

1. α is a quasi-idempotent: α/= α2 = α4;\
2. The digraph Γ satisfies:
   a. Every vertex v ∈ V is either stationary (α(v) = v) or maps to a stationary vertex;
   b. All non-trivial strongly connected components are 2-cycles;
   c. There is no directed path v1 → v2 → v3 such that all of v1, v2, v3 are non- stationary.

Proof. (1) ⇒ (2): If α ∈ On is quasi-idempotent, then from the algebraic characterization in the literature, each non-stationary block maps into a stationary block. Therefore, each non-stationary vertex maps to a stationary vertex (a). Since no sequence of such mappings leads to longer non-trivial cycles, all non-trivial SCCs must be 2-cycles (b). Longer directed paths among non-stationary vertices would contradict this (c).
(2) ⇒ (1): If the digraph structure satisfies (a)–(c), then α2 maps each non- stationary point to the image of a stationary vertex. Since stationary points are fixed, α2 = α4, and since α/= α2 (due to non-stationarity), α is quasi-idempotent.
Example 3.4. *Let α ∈ O11 be defined by:*

$$\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} & (9,10) & \{11\} \\ 1 & 2 & 5 & 6 & 9 & 10 \end{pmatrix}$$

We analyze the structure of α:
The blocks {1, 2}, {5, 6}, {9, 10} are stationary since the image lies within the same block.
The blocks {3, 4}, {7, 8}, {11} are non-stationary since they map to other values.
Each non-stationary block maps into a stationary one: {5, 6} '→ 5 ∈ {5, 6}, and similarly others map to 6 and 9, making them eventually stationary under squaring.
Compute α^2:

$$\alpha^2 = \begin{pmatrix} \{1,2,3,4\} & \{5,6,7,8\} & \{9,10,11\} \\ 1 & 5 & 9 \end{pmatrix}$$

Here, α2 is idempotent, so α2 = α4, and since α≠α^2, it follows that α is quasi- idempotent.

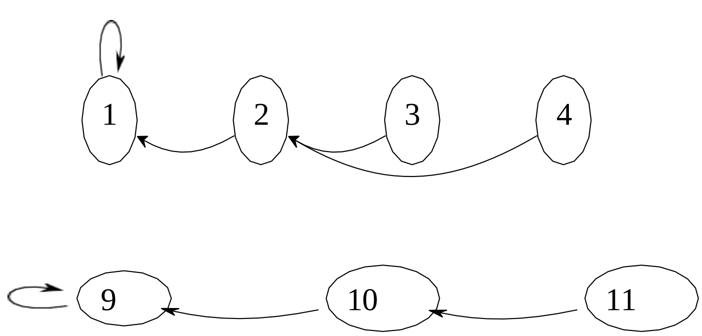

Fig. 1. Functional digraph of the transformation α ∈ O11

Note: All directed edges from non-stationary vertices terminate at stationary ones, satisfying the structural characterization.

Proposition 3.5. Let $\Gamma \in On$ be a quasi-idempotent digraph. Then Γ decomposes into:

Stationary vertices: $\{v \in V \mid \alpha(v) = v\}$,

Disjoint 2-cycles: $\{v \leftrightarrow w \mid \alpha(v) = w, \alpha(w) = v, v \neq w\}$.

Proof. By Theorem 3.3, only self-loops (stationary) and mutual pairings (2-cycles) are allowed. Since longer paths or cycles would violate quasi-idempotency, the structure must consist of such disjoint components.

Lemma 3.6. In any quasi-idempotent digraph $\Gamma \in On$:

The maximum length of any directed path is 1;

There is no directed path $x \to y \to z$ with x, y, z all non-stationary.

Proof. Since non-stationary vertices must point to stationary ones, no chain of length greater than 1 is possible among them. Any violation of this would imply a non-trivial path beyond the constraints in Theorem 3.1.

4. CONCLUSION

In this work, we extended the algebraic study of quasi-idempotent in the semigroup On of full order-preserving transformations by introducing a graph-theoretic perspective. Using functional digraphs, we established a necessary and sufficient condition for a transformation to be quasi-idempotent based on the structure of its directed graph. Our main results reveal that such a transformation admits a decomposition into fixed points and disjoint 2-cycles, with no directed path connecting three dis- tinct non-stationary vertices. These findings not only reinforce the algebraic conditions previously established but also offer a more intuitive, visual interpretation of quasi- idempotent behavior in On. This approach lays the groundwork for future research in the combinatorial and algorithmic analysis of transformation semigroups. In particular, it opens up possibilities for developing efficient detection and generation algorithms for quasi-idempotent using graph traversal techniques. It may also provide insights into broader classes of semigroups where graphical representations can illuminate otherwise complex structural relationships.

Conflicts Of Interest

The authors declare that they have no competing interests.

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