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Research Article

Approximate Solution of Two-Dimensional Linear Systems of Fractional Order Partial Integro-Differential Equations Using Variational Iteration Method

Batool I. Asker[1], Fadhel S. Fadhel[1,*], Mohammed S. Ismael[2], Samsul Ariffin Abdul Karim[3]

[1] *Department of Mathematics and Computer Applications, College of Sciences, Al-Nahrain University, Jadriya, Baghdad, Iraq*

[2] *Department of Mathematics, College of Sciences, University of Baghdad, Baghdad, Iraq*

[3] *Institute of Strategic Industrial Decision Modellil9ol-['ng (ISIDM), School of Quantitative Sciences, UUM College of Arts & Sciences, Universiti Utara Malaysia, 06010 Sintok Kedah Darul Aman, Malaysia*

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ABSTRACT

The main objective of this paper is to introduce and solve two-dimension linear system of fractional order partial integro-differential equations using the most well-known approximation methods, which is the variational iteration method. After that, the convergence of the derived sequence of iterative approximations to the exact solution is proved, which is must be assumed to be exist according to existence and uniqueness theorem of partial integro-differential equations. Two illustrative examples are considered utilizing the computer software (PTC Mathcad) and then comparison between the exact and the approximate solutions for a test examples are given to show the efficiency and reliability of the proposed method.

1. INTRODUCTION

Fractional calculus explores the concepts of differentiation and integration non-integer order. The fractional calculus subject is more general version of the classical analysis of calculus. Fractional calculus becomes nowadays more popular than classical calculus due to its implementation in many fields, such as science and technology. Also, fractional order partial integro-differential equations (FPIDEs) are very popular due to its excellent simulation properties various scientific fields. It is used to represent physical and engineering phenomena's that are largely described by fractional differential equations, in which fractional derivation models are used to better identify those systems that require accurate modeling attenuation. Non-Fourier conduction, sound dissipation, geophysics, relaxation, creep, viscoelasticity, rheology, fluid dynamics, COVID-19 and malaria are an applications models of such to topic [1-3].
Fractional integro-differential equations are a special type of equations having the integral equations with either fractional derivative. In recent years, there has been a growing interest the concerning integro-differential equations, in different topics, such as nonlinear functional analysis and their applications in the theory of engineering, mechanics, chemistry, physics, economics, kinetics, astronomy, biology, potential theory and electro statistics include integro-differential equations, [4,5].
The FPIDEs are a generalization of the classical integer order partial differential equations (PDEs) and integral equations, which are increasingly used to pattern problems in fluid flow, finance and other areas of applications. Moreover, fractional derivatives provide an excellent writing for the characterization of memory and inherited properties of various problems and operations, [6].
Among the most popular semi-analytical methods is the variational iteration method (VIM), which is a powerful iterative approximated method based on the Lagrange multiplier technique. Since its emergence in the late 1990s, where it has been widely used to solve various problems, including initial value problems for fractional differential equations, [7-16]. In addition to the above, the VIM has demonstrated remarkable reliability and efficiency in a wide range of scientific

*Corresponding author. Email: fsf@sc.nahrainuniv.edu.iq

applications, both linear and nonlinear. It was shown by many authors that this method is more powerful than some other existing techniques, such as the Adomian decomposition method, perturbation method, etc. One of the main advantages of the VIM is that it provides easily applicable successive approximations that converge rapidly towards the exact solution, [17].

In this article, two-dimensional systems of FPIDEs will be solved approximately using the VIM as an iteration approach and then prove the convergence of the obtained sequence of iterative solutions to the exact solution, which must be exist virtually based on the existence and uniqueness theorem of the solution of such type of problems.

2. PRELIMINARIES

In this section, some basic definitions and properties of fractional order derivatives and integrals related to the present work will be given for completeness purpose. In this section, it will be assumed that $C_t^m([a,b]\times[0,T])$ to be the space of all m-continuously differentiable functions up to order m with respect to t.

Definition 2.1, [18]. The left Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function u is defined by:

$$ {}_0^RI_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\,u(s)ds,\ \ s>0 $$

where $m-1<\alpha\le m, m\in\mathbb{N}, \Gamma(\alpha)$ is the classical gamma function and ${}^RI_x^0u(x)=u(x)$.

Definition 2.2, [19]. The left-differential Riemann-Liouville fractional operator of order $\alpha > 0$ of a function u is defined as:

$$ {}_a^RD_x^\alpha u(x) = \frac{1}{\Gamma(m-1)}\frac{d^m}{dx^m}\int_a^x (x-s)^{m-\alpha-1}\,u(s)ds $$

where $m-1<\alpha\le m, m\in\mathbb{N}$.

Definition 2.3, [20,21]. The Caputo fractional order derivative of a function u of order $\alpha > 0$, is defined as:

$$ {}_0^CD_x^\alpha u(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1}\,u^{(m)}(s)ds $$

where $m-1<\alpha\le m, m\in\mathbb{N}$

Some properties of fractional order derivatives and integrals of order $\alpha>0$, where $m-1<\alpha\le m, m\in\mathbb{N}$, which are needed later on in this work are stated below [19]:

i. ${}^CD_x^\alpha\ {}^RI_x^\alpha u(x) = u(x)$.

ii. ${}^RI_x^\alpha\ {}^CD_x^\alpha u(x) = u(x) - \sum_{k=0}^{n-1} u^{(k)}(0^+)\frac{u^k}{k!}$, $u>0$.

where 0^+ refers to the right-hand sided limit of the function as x tends to 0.

iii. ${}^RI_x^\alpha u^v = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)}u^{v+\alpha}, v>-1, u>0$.

iv. ${}^CD_x^\alpha \sum_{i=0}^k c_iu_i(x) = \sum_{i=0}^k c_i\,{}^CD_x^\alpha u_i(x)$, where c_i is a constant, $\forall\ i=0,1,\cdots,k$.

3. VARIATIONAL ITERATION METHOD FOR SOLVING SYSTEMS OF FPIDEs

In this section, the VIM will be generalized to solve systems of FPIDEs, in which the general problem formulation is to solve the system:

$$ \begin{aligned} {}_0^CD_t^{\alpha_1}u_1(x,t) &= g_1(x,t) + {}_aI_x^{\beta_1}\,{}_0I_t^{\gamma_1}k_1(x,t,u_1(x,t),u_2(x,t),\dots,u_k(x,t)) \\ {}_0^CD_t^{\alpha_2}u_2(x,t) &= g_2(x,t) + {}_aI_x^{\beta_2}\,{}_0I_t^{\gamma_2}k_2(x,t,u_1(x,t),u_2(x,t),\dots,u_k(x,t)) \\ &\ \vdots \\ {}_0^CD_t^{\alpha_k}u_k(x,t) &= g_k(x,t) + {}_aI_x^{\beta_k}\,{}_0I_t^{\gamma_k}k_k(x,t,u_1(x,t),u_2(x,t),\dots,u_k(x,t)) \end{aligned} \tag{1} $$

with initial conditions:

$$ u_1(x,0)=c_1, u_2(x,0)=c_2,\dots,u_k(x,0)=c_k \tag{2} $$

for all $k\in\mathbb{N}$, $\alpha_p,\beta_p,\gamma_p\in(0,1], \forall\ p=1,2,\cdots,k$ and $(x,t)\in\Omega$, where $\Omega=\{(x,t)\in\mathbb{R}^2|\ a\le x\le b, 0\le t\le T\}$ and k_p are the kernel functions which are given, g_p are given continuous functions, u_p are an unknown real valued functions to be evaluated.

The variational iteration formulation for solving system (1) starts by introducing first the next theorem, which is necessary for obtaining the approximation-numerical results, [22].

Theorem 3.1, [22]. Consider the generalized system of FPIDEs (1) defined over the region Ω and suppose that the kernel functions k_p satisfies Lipschitz condition with respect to $u_{p,n}$ and Lipschitz constants L_p, such that $L_p < \frac{\Gamma(w_p+1)\Gamma(\beta_p+1)}{T^{w_p}(b-a)^{\beta_p}}, \forall\, p = 1,2,\cdots,k$. Then Eq. (1) has a unique solution.

The sequence of iterative approximate solutions using the proposed approach is derived in the next theorem:

Theorem 3.2. Consider the generalized system of FPIDE (1) with the initial conditions given in Eqs. (2), which has a unique solution and $u_{p,n} \in C_t^m([a,b]\times[0,T])$ be the approximate solution of system (1). Then the sequence of approximate solutions using the VIM are approximated by:

$$u_{p,n+1}(x,t) = u_{p,n}(x,t) - {}_0I_t^{\alpha_p}\{{}_0^CD_t^{\alpha_p}\, u_{p,n}(x,t) - g_p(x,t) - {}_a^RI_x^{\beta_p}\,{}_0^RI_t^{\gamma_p}\, k_p(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\dots,u_{p,n}(y,s)\} \quad (3)$$

for all $n = 0,1,2,\dots,\ \ p = 1,2,\dots,k$.

Proof. For any $p = 1,2,\dots,k$; multiply both sides of the p^{th} equation of system (1) by a general Lagrange multiplier λ_p, which will yield to:

$$\lambda_p\{{}_0^CD_t^{\alpha_p}u_p(x,t) - g_p(x,t) - {}_a^RI_x^{\beta_p}\,{}_0^RI_t^{\gamma_p}\,k_p(x,t,u_1(x,t),u_2(x,t),\dots,u_p(x,t))\} = 0 \quad (4)$$

where $\alpha_p,\beta_p,\gamma_p \in (0,1]$.

Now, take the left-hand sided Reiman-Liouville fractional order integral ${}_0^RI_t^{\alpha_p}$ of both sides of Eq. (4), implies to:

$${}_0^RI_t^{\alpha_p}\lambda_p\{{}_0^CD_t^{\alpha_p}u_p(x,t) - g_p(x,t) - {}_a^RI_x^{\beta_p}\,{}_0^RI_t^{\gamma_p}\,k_p(x,t,u_1(x,t),u_2(x,t),\dots,u_p(x,t))\} \quad (5)$$

hence the correction functional for the p^{th} equation of system (1) will take the form:

$$u_{p,n+1}(x,t) = u_{p,n}(x,t) + {}_0^RI_t^{\alpha_p}\lambda_p(x,s)\Big\{{}_0^CD_s^{\alpha_p}\, u_{p,n}(x,s) - g_p(x,s) - {}_a^RI_x^{\beta_p}\,{}_0^RI_t^{\gamma_p}\,k_p(y,s,\tilde{u}_{1,n}(y,s),\tilde{u}_{2,n}(y,s),\dots,\tilde{u}_{p,n}(y,s)\Big\} \quad (6)$$

where $\tilde{u}_{1,n},\tilde{u}_{2,n},\dots,\tilde{u}_{k,n}$ are considered as restricted variations. Hence, the approximate solution of the correction functional is:

$$u_{p,n+1}(x,t) = u_{p,n}(x,t) + {}_0^RI_t^{\alpha}\lambda_p(x,s)\Big\{D_t u_{p,n}(x,s) - g_p(x,s) - {}_a^RI_x^{\beta_p}\,{}_0^RI_t^{\gamma_p}\,k_p(y,s,\tilde{u}_{1,n}(y,s),\tilde{u}_{2,n}(y,s),\dots,\tilde{u}_{p,n}(y,s)\Big\} \quad (7)$$

Therefore, carrying out the first variations of both sides of Eq. (7) relative to $u_{1,n},u_{2,n},\dots,u_{k,n}$, respectively using the assumptions that:

$\delta u_{1,n}(x,0) = 0, \delta u_{2,n}(x,0) = 0, \dots, \delta u_{k,n}(x,0) = 0$ and $\delta g_p(x,t) = 0$

Now, because it is so difficult to evaluate λ_p for the correction functional (7), the fractional integral ${}_0^RI_t^{\alpha_p}$ is approximated by a single integral and hence,

$$\delta u_{p,n+1}(x,t) = \delta u_{p,n}(x,t) + \int_0^t D_s\,\lambda_p(x,s)\delta u_{p,n}(x,s)ds \quad (8)$$

Thus, upon using integration by parts with respect to s, we get:

$$\delta u_{p,n+1}(x,t) = \delta u_{p,n}(x,t) + \lambda_p(x,s)\delta u_{p,n}(x,s)|_{\,s=t} + \int_0^t \delta u_{p,n}(x,s)\lambda_p'(x,s)ds$$

Therefore:

$$\delta u_{p,n+1}(x,t) = \left(1+\lambda_p\right)\delta u_{p,n}(x,s)|_{\,s=t} - \int_0^t \delta u_{p,n}(x,s)\lambda_p'(x,s)ds \quad (9)$$

Based on variational theory, the following necessary condition is obtained as a result for an arbitrary $\delta u_{p,n}$

$$\lambda_p'(x,s) = 0, \forall\, p = 1,2,\cdots,k \quad (10)$$

with initial condition:

$$1 + \lambda_p(x,s)\,|_{s=t} = 0 \quad (11)$$

Solving Eqs. (10) with initial conditions (11), the following general Lagrange multipliers are obtained $\lambda_p(x,s) = -1$, for all $p = 1,2,\dots,k$.

Hence, after substituting the values of λ_p back into the correction functional (6), the following variational iteration formula is obtained:

$$u_{p,n+1}(x,t) = u_{p,n}(x,t) - {}_0I_t^{\alpha_p}\left\{{}_0^CD_t^{\alpha_p}u_{p,n}(x,s) - g_p(x,s) - {}_aI_x^{\beta_p}\,{}_0I_t^{\gamma_p}k_p(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\dots,u_{p,n}(y,s)\right\}$$

for all $p = 1,2,\dots,k$.

4. CONVERGENCE ANALYSIS

The convergence of the obtained iterative sequence of approximate solutions of system (1) to the exact solution may be achieved and proved as in the next theorem:

Theorem 4.1. Let $u_p, u_{p,n} \in C_t^m([a,b]\times[0,T])$ be respectively the exact and approximate solutions of system (1) and (3). If $E_{p,n}(x,t) = u_{p,n}(x,t) - u_p(x,t)$ and the kernels k_p satisfies Lipschitz condition with constants $L_p < \frac{\Gamma(\alpha_p+1)\Gamma(\beta_p+1)}{T^{\alpha_p}(b-a)^{\beta_p}}$, then the sequence of approximate solutions $\{u_{p,n}\}, n = 0,1,2,\ldots$ converge to the exact solution $u_p(x,t)$, where $p = 1,2,\ldots,k;\ k \in \mathbb{N}, \alpha_p, \beta_p, \gamma_p \in (0,1]$.

Proof. Using Eq. (3), the approximate solutions of Eq. (1) obtained using the VIM is given by:

$$u_{p,n+1}(x,t) = u_{p,n}(x,t) - {}_0I_t^{\alpha_p}\left\{{}_0^cD_t^{\alpha_p}u_{p,n}(x,s) - g_p(x,s) - {}_aI_x^{\beta_p}\,{}_0I_t^{\gamma_p}k_p\big(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\ldots,u_{p,n}(y,s)\big)\right\}$$

Also, u_p is the exact solution of Eq. (1) and hence, it satisfies Eq. (3), i.e.,

$$u_p(x,t) = u_p(x,t) - {}_0I_t^{\alpha_p}\left\{{}_0^cD_t^{\alpha_p}u_{p,n}(x,s) - g_p(x,s) - {}_0I_x^{\beta_p}\,{}_0I_t^{\gamma_p}k_p\big(y,s,u_1(y,s),u_2(y,s),\ldots,u_p(y,s)\big)\right\}$$

Subtracting u_p from $u_{p,n+1}$ implies to:

$$\begin{aligned}u_{p,n+1}(x,t) - u_p(x,t) = {}& u_{p,n}(x,t) - u_p(x,t) - {}_0I_t^{\alpha_p}\Big\{{}_0^cD_t^{\alpha_p}u_{p,n}(x,s) - {}_0^cD_t^{\alpha_p}u_p(x,s) - g_p(x,s) + g_p(x,s) - \\ & {}_aI_x^{\beta_p}\,{}_0I_t^{\gamma_p}\big[k_p\big(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\ldots,u_{p,n}(y,s)\big) - k_p\big(y,s,u_1(y,s),u_2(y,s),\ldots,u_p(y,s)\big)\big]\Big\}\end{aligned}$$

and hence:

$$\begin{aligned}E_{p,n+1}(x,t) &= E_{p,n}(x,t) - {}_0I_t^{\alpha_p}\Big\{{}_0^cD_t^{\alpha_p}E_{p,n}(x,s) - {}_aI_x^{\beta_p}\,{}_0I_t^{\gamma_p}\big[k_p\big(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\ldots,u_{p,n}(y,s)\big) - \\ &\qquad k_p\big(y,s,u_1(y,s),u_2(y,s),\ldots,u_p(y,s)\big)\big]\Big\} \\ &= E_{p,n}(x,t) - E_{p,n}(x,t) + E_{p,n}(x,0) \\ &\qquad + {}_0I_t^{\alpha_p}\,{}_aI_x^{\beta_p}\,{}_0I_t^{\gamma_p}\big\{k_p\big(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\ldots,u_{p,n}(y,s)\big) \\ &\qquad - k_p\big(y,s,u_1(y,s),u_2(y,s),\ldots,u_p(y,s)\big)\big\}\end{aligned}$$

and since $E_{p,n}(x,0) = 0$, therefore when $w_p = \alpha_p + \gamma_p$, we will get:

$$E_{p,n+1}(x,t) = {}_aI_x^{\beta_p}\,{}_0I_t^{w_p}\big\{k_p\big(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\ldots,u_{p,n}(y,s)\big) - k_p\big(y,s,u_1(y,s),u_2(y,s),\ldots,u_p(y,s)\big)\big\} \tag{12}$$

Since the kernel functions k_p satisfies Lipschitz condition with constants L_p and upon applying the supremum norm on Eq. (12), getting:

$$\begin{aligned}\|E_{p,n+1}(x,t)\| &\le {}_aI_x^{\beta_p}\,{}_0I_t^{w_p}\big\|k_p\big(y,s,u_{1,n}(y,s),u_{2,n}(y,s),\ldots,u_{p,n}(y,s)\big) - k_p\big(y,s,u_1(y,s),u_2(y,s),\ldots,u_p(y,s)\big)\big\| \\ &\le L_p\,{}_aI_x^{\beta_p}\,{}_0I_t^{w_p}\big\|\big(u_{1,n}(y,s),u_{2,n}(y,s),\ldots,u_{p,n}(y,s)\big) - \big(u_1(y,s),u_2(y,s),\ldots,u_p(y,s)\big)\big\| \\ &\le L_p\,{}_aI_x^{\beta_p}\,{}_0I_t^{w_p}\big(\|E_{1,n}(y,s)\| + \|E_{2,n}(y,s)\| + \cdots + \|E_{p,n}(y,s)\|\big)\end{aligned} \tag{13}$$

Using the definition of left-hand sided Riemann-Liouville of fractional integral twice in inequality (13), then inequality (13) will be reduced to:

$$\|E_{p,n+1}(x,t)\| \le \frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t (t-s)^{w_p-1}\int_a^x (x-y)^{\beta_p-1}\big(\|E_{1,n}(y,s)\| + \|E_{2,n}(y,s)\| + \cdots + \|E_{p,n}(y,s)\|\big)dyds \tag{14}$$

$$\le \frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t\int_a^x (x-y)^{\beta_p-1}(t-s)^{w_p-1}\big(\|E_{1,n}(y,s)\| + \|E_{2,n}(y,s)\| + \cdots + \|E_{p,n}(y,s)\|\big)\,dyds$$

$$\le \frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t\int_a^x (x-a)^{\beta_p-1}\,t^{w_p-1}\big(\|E_{1,n}(y,s)\| + \|E_{2,n}(y,s)\| + \cdots + \|E_{p,n}(y,s)\|\big)\,dyds \tag{15}$$

Thus, if $n = 0$, then:

$$\begin{aligned}\|E_{p,1}(x,t)\| &\le \frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t\int_a^x (x-a)^{\beta_p-1}\,t^{w_p-1}\big(\|E_{1,0}(y,s)\| + \|E_{2,0}(y,s)\| + \cdots + \|E_{p,0}(y,s)\|\big)dyds \\ &= \frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\frac{(x-a)^{\beta_p}}{\beta_p}\frac{t^{w_p}}{w_p}\big(\|E_{1,0}(y,s)\| + \|E_{2,0}(y,s)\| + \cdots + \|E_{p,0}(y,s)\|\big) \\ &\le \frac{L_p(x-a)^{\beta_p}t^{w_p}}{\Gamma(w_p+1)\Gamma(\beta_p+1)}\big(\|E_{1,0}(y,s)\| + \|E_{2,0}(y,s)\| + \cdots + \|E_{p,0}(y,s)\|\big)\end{aligned} \tag{16}$$

If $n = 1$, then:

$$\|E_{p,2}(x,t)\| \le \frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t\int_a^x (x-a)^{\beta_p-1}\,t^{w_p-1}\big(\|E_{1,1}(y,s)\| + \|E_{2,1}(y,s)\| + \cdots + \|E_{p,1}(y,s)\|\big)dyds$$

$$\le \frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t\int_a^x (x-a)^{\beta_p-1}\ t^{w_p-1}\frac{L_p(x-a)^{\beta_p}t^{w_p}}{\Gamma(w_p+1)\Gamma(\beta_p+1)}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right)dyds$$
$$=\frac{L_p^2}{\Gamma(w_p)\Gamma(\beta_p)\Gamma(w_p+1)\Gamma(\beta_p+1)}\int_0^t\int_a^x (x-a)^{2\beta_p-1}\ t^{2w_p-1}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right)dyds$$
$$=\frac{L_p^2}{\Gamma(w_p)\Gamma(\beta_p)\Gamma(w_p+1)\Gamma(\beta_p+1)}\frac{(x-a)^{2\beta_p}}{2\beta_p}\frac{t^{2w_p}}{2w_p}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right)$$
$$\le\left(\frac{L_p}{2\Gamma(w_p+1)\Gamma(\beta_p+1)}\right)^2(x-a)^{2\beta_p}t^{2w_p}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right) \tag{17}$$

If $n=2$, then:

$$\|E_{p,3}(x,t)\|\le\frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t\int_a^x (x-a)^{\beta_p-1}\ t^{w_p-1}\left(\|E_{1,2}(y,s)\|+\|E_{2,2}(y,s)\|+\cdots+\|E_{p,2}(y,s)\|\right)dyds$$
$$\le\frac{L_p}{\Gamma(w_p)\Gamma(\beta_p)}\int_0^t\int_a^x (x-a)^{\beta_p-1}\ t^{w_p-1}\left(\frac{L_p}{2\Gamma(w_p+1)\Gamma(\beta_p+1)}\right)^2(x-a)^{2\beta_p}t^{2w_p}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right)dyds$$
$$=\frac{L_p^3}{2^2\Gamma(w_p)\Gamma(\beta_p)\Gamma^2(w_p+1)\Gamma^2(\beta_p+1)}\int_0^t\int_a^x (x-a)^{3\beta_p-1}\,t^{3w_p-1}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right)dyds$$
$$\le\frac{L_p^3}{2^2 3^2\Gamma^3(w_p+1)\Gamma^3(\beta_p+1)}(x-a)^{3\beta_p}t^{3w_p}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right) \tag{18}$$

Therefore, for any arbitrary natural number n, and upon mathematical induction, it can be concluded that:

$$\|E_{p,n+1}(x,t)\|\le\frac{L_p^n(x-a)^{n\beta_p}t^{nw_p}}{(2\times3\times\ldots\times n)^2\Gamma^n(w_p+1)\Gamma^n(\beta_p+1)}\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right)$$
$$=\frac{1}{(n!)^2}\left(\frac{L_p(b-a)^{\beta_p}T^{w_p}}{\Gamma(w_p+1)\Gamma(\beta_p+1)}\right)^n\left(\|E_{1,0}(y,s)\|+\|E_{2,0}(y,s)\|+\cdots+\|E_{p,0}(y,s)\|\right)$$

Now, since $L_p<\frac{\Gamma(w_p+1)\Gamma(\beta_p+1)}{T^{w_p}(b-a)^{\beta_p}}$ and hence as $n\to\infty$, then $\left(\|E_{1,n}(y,s)\|+\|E_{2,n}(y,s)\|+\cdots+\|E_{p,n}(y,s)\|\right)\to0$, and thus $u_{p,n}(x,t)\to u_p(x,t)$ as $n\to\infty$, which means that the sequence of approximate solutions of Eq. (3) converge to the exact solution of Eq. (1).

5. ILLUSTRATIVE EXAMPLES

In this section, two illustrative test examples will be considered and simulated using the VIM (3). The considered examples are for linear case.

Example 5.1. Consider the problem of solving the following linear system of FPIDEs:

$${}_0^CD_t^{\alpha_1}u_1(x,t)=-0.64784\,t^{2.33}\,x^{3.6}+1.11917\,t^{0.6}\,x^2+{}_aI_x^{\beta_1}\,{}_0I_t^{\gamma_1}\{(xt)u_2(x,t)\} \tag{19}$$
$${}_0^CD_t^{\alpha_2}u_2(x,t)=2.25675t^{0.5}x^2-0.26192t^{1.45}x^{3.8}+0.27083t^{2.45}x^{2.8}+{}_aI_x^{\beta_2}\,{}_0I_t^{\gamma_2}\{(x-t)u_1(x,t)\} \tag{20}$$

with initial conditions:

$$u_1(x,0)=u_2(x,0)=0 \tag{21}$$

where $\alpha_1=0.4,\alpha_2=0.5,\beta_1=0.6,\beta_2=0.8,\gamma_1=0.33$ and $\gamma_2=0.45$, for all $(x,t)\in[0,1]\times[0,1]$. For comparison purpose, the exact solutions are given by $u_1(x,t)=x^2t$ and $u_2(x,t)=2x^2t$. Now, by applying the VIM, and by using with the initial approximate solution as follows:

$$u_{1,0}(x,t)=-0.64784\,t^{2.33}\,x^{3.6}+1.11917\,t^{0.6}\,x^2$$

and

$$u_{2,0}(x,t)=2.25675t^{0.5}x^2-0.26192t^{1.45}x^{3.8}+0.27083t^{2.45}x^{2.8}$$

Then, the first and second approximate solutions of Eq. (19) which are denoted by $u_{1,1}(x,t)$ and $u_{1,2}(x,t)$, respectively while the first and second approximate solutions of Eq. (20) which are denoted by $u_{2,1}(x,t),u_{2,2}(x,t)$, are evaluated. The approximate solutions are computed for $x=0.5$, $t\in[0,1],\Delta t=0.1$. Also, comparison is then made with the exact solutions, where the values are listed in Table 5.1. From the results of Table I, the convergence and the accuracy of the obtained results between the exact and approximate solutions may be seen.

Also, Table II presents for comparison purpose the absolute errors between the approximate and the exact solutions for different values of x and t.

TABLE I. THE EXACT AND APPROXIMATE SOLUTIONS OF EXAMPLE 5.1

t	$u_1(x,t)$	$u_{1,1}(x,t)$	$u_{1,2}(x,t)$	$u_2(x,t)$	$u_{2,1}(x,t)$	$u_{2,2}(x,t)$
0	0	0	0	0	0	0
0.1	0.025	0.025197	0.025	0.05	0.050297	0.05
0.2	0.05	0.0508	0.050002	0.1	0.100623	0.100003
0.3	0.075	0.076742	0.075006	0.15	0.150822	0.150008
0.4	0.1	0.102937	0.10001	0.2	0.200863	0.200014
0.5	0.125	1.29301	0.125014	0.25	0.250755	0.250016
0.6	0.15	0.155752	0.150016	0.3	0.300532	0.300012
0.7	0.175	1.82216	0.175013	0.35	0.350248	0.349996
0.8	0.2	0.208621	0.200005	0.4	0.39997	0.399966
0.9	0.225	0.234906	0.224991	0.45	0.44978	0.449919
1	0.25	0.261012	0.249976	0.5	0.499776	0.499853

TABLE II. THE ABSOLUTE ERRORS OF EXAMPLE 5.1.

t	$\lvert u_1(x,t)-u_{1,1}(x,t)\rvert$	$\lvert u_1(x,t)-u_{1,2}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,1}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,2}(x,t)\rvert$
0	0	0	0	0
0.1	1.972×10^{-4}	2.767×10^{-7}	2.975×10^{-4}	4.824×10^{-7}
0.2	8.003×10^{-4}	1.99×10^{-6}	6.229×10^{-4}	3.189×10^{-6}
0.3	1.742×10^{-3}	5.528×10^{-6}	8.221×10^{-4}	8.199×10^{-6}
0.4	2.937×10^{-3}	1.01×10^{-5}	8.632×10^{-4}	1.358×10^{-5}
0.5	4.301×10^{-3}	1.411×10^{-5}	7.552×10^{-4}	1.606×10^{-5}
0.6	5.752×10^{-3}	1.559×10^{-5}	5.325×10^{-4}	1.165×10^{-5}
0.7	7.216×10^{-3}	1.275×10^{-5}	2.478×10^{-4}	3.701×10^{-6}
0.8	8.621×10^{-3}	4.576×10^{-6}	3.044×10^{-4}	3.362×10^{-5}
0.9	9.906×10^{-3}	8.577×10^{-6}	2.196×10^{-4}	8.079×10^{-5}
1	0.011	2.428×10^{-5}	2.243×10^{-4}	1.465×10^{-4}

Similarly, if we choose other values for $\alpha_1$and α_2 such as $\alpha_1, \alpha_2 = 1$, and substitute this value in Eq. (19), we can get the value of $g(x,t)$ and Eq. (19) will be:

$$ {}_0^CD_t^{\alpha_1} u(x,t) = -0.81637\, t^{2.75}\, x^{2.5} + 1.12837\, \sqrt{t}\, x + {}_aI_x^{\beta_1}\, {}_0I_t^{\gamma_1}\{(xt)u_2(x,t)\} \tag{22} $$

$$ {}_0^CD_t^{\alpha_2} u_2(x,t) = -0.53975 t^{2.3}\, x^{3.25} + 2.14734\, t^{0.8}\, x + {}_aI_x^{\beta_2}\, {}_0I_t^{\gamma_2}\{(x-t)u_1(x,t)\} \tag{23} $$

with the initial condition

$$ u_1(x,0) = 0, u_2(x,0) = 0 \tag{24} $$

where,

$\alpha_1, \alpha_2 = 1, \beta_1 = 0.6, \beta_2 = 0.8, \gamma_1 = 0.33$, and $\gamma_2 = 0.45$ for all $(x,t) \in [0,1] \times [0,1]$. For comparison purpose, the exact solutions are given by $u_1(x,t) = x^2 t$ and $u_2(x,t) = 2x^2 t$. Now, by applying the VIM, and by using with the initial approximate solution as follows:

$$ u_{1,0}(x,t) = -0.81637\, t^{2.75}\, x^{2.5} + 1.12837\, \sqrt{t}\, x $$

and

$$ u_{2,0}(x,t) = -0.53975 t^{2.3}\, x^{3.25} + 2.14734\, t^{0.8}\, x $$

Then, the first and second approximate solutions of Eq. (25) which are denoted by $u_{1,1}(x,t)$ and $u_{1,2}(x,t)$, respectively while the first and second approximate solutions of Eq. (26) which are denoted by $u_{2,1}(x,t), u_{2,2}(x,t)$, are evaluated. The approximate solutions are computed for $x = 0.5, t \in [0,1], \Delta t = 0.1$. Also, comparison is then made with the exact solutions where the values are listed in Table III. From the results of Table III, the convergence and the accuracy of the obtained results between the exact and approximate solutions may be seen.

Also, Table IV, presents for comparison purpose the absolute errors between the approximate and the exact solutions for different values of x and t.

TABLE III. THE EXACT AND APPROXIMATE SOLUTIONS OF EXAMPLE 5.1

t	$u_1(x,t)$	$u_{1,1}(x,t)$	$u_{1,2}(x,t)$	$u_2(x,t)$	$u_{2,1}(x,t)$	$u_{2,2}(x,t)$
0	0	0	0	0	0	0
0.1	0.025	0.024992	0.025	0.05	0.049977	0.05
0.2	0.05	0.049924	0.05	0.1	0.099894	0.1
0.3	0.075	0.074707	0.075	0.15	0.149772	0.15
0.4	0.1	0.099237	0.099999	0.2	0.199658	0.200001
0.5	0.125	0.1234	0.124998	0.25	0.249618	0.250003
0.6	0.15	0.147072	0.149998	0.3	0.299737	0.300009
0.7	0.175	0.170126	0.174998	0.35	0.350113	0.350024
0.8	0.2	0.192428	0.200001	0.4	0.40086	0.400053
0.9	0.225	0.213841	0.255011	0.45	0.452106	0.450107
1	0.25	0.234228	0.250035	0.5	0.503995	0.500199

TABLE IV. THE ABSOLUTE ERRORS OF EXAMPLE 5.1.

t	$\lvert u_1(x,t)-u_{1,1}(x,t)\rvert$	$\lvert u_1(x,t)-u_{1,2}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,1}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,2}(x,t)\rvert$
0	0	0	0	0
0.1	7.56×10^{-6}	1.713×10^{-9}	2.232×10^{-5}	1.251×10^{-10}
0.2	7.607×10^{-5}	4.034×10^{-8}	1.061×10^{-4}	7.101×10^{-9}
0.3	2.933×10^{-4}	2.332×10^{-7}	2.282×10^{-4}	1.228×10^{-7}
0.4	7.63×10^{-4}	7.352×10^{-7}	3.422×10^{-4}	7.763×10^{-7}
0.5	1.6×10^{-3}	1.587×10^{-6}	3.815×10^{-4}	3.099×10^{-6}
0.6	2.928×10^{-3}	2.464×10^{-6}	2.626×10^{-4}	9.417×10^{-6}
0.7	4.874×10^{-3}	2.339×10^{-6}	1.133×10^{-4}	2.386×10^{-5}
0.8	7.572×10^{-3}	9.619×10^{-7}	8.601×10^{-4}	5.304×10^{-5}
0.9	0.011	1.128×10^{-5}	2.106×10^{-3}	1.069×10^{-4}
1	0.016	3.478×10^{-5}	3.995×10^{-3}	1.995×10^{-4}

Example 5.2. Consider the problem of solving the following linear system of FPIDEs:

$$ {}_0^C D_t^{\alpha_1} u_1(x,t) = -0.81637\, t^{2.75}\, x^{2.5} + 1.12837\, t^{0.5}\, x + {}_a I_x^{\beta_1}\, {}_0 I_t^{\gamma_1}\left[(xt)\left(u_1(x,t) + u_2(x,t)\right)\right] \tag{25} $$

$$ {}_0^C D_t^{\alpha_2} u_2(x,t) = -0.50499\, t^{2.3} x^{3.3} + 2.25675\, t^{0.5}\, x + {}_a I_x^{\beta_2}\, {}_0 I_t^{\gamma_2}\left[(x^2 t) u_1(x,t)\right] \tag{26} $$

with initial conditions:

$$ u_1(x,0) = u_2(x,0) = 0 \tag{27} $$

where $\alpha_1 = 0.5, \alpha_2 = 0.2, \beta_1 = 0.5, \beta_2 = 0.25, \gamma_1 = 0.75$ and $\gamma_2 = 0.3$, for all $(x,t) \in [0,1]\times[0,1]$. For comparison purpose, the exact solutions are given by $u_1(x,t) = xt$ and $u_2(x,t) = 2xt$. Now, by applying the VIM, and by using with the initial approximate solution defined by:

$$ u_{1,0}(x,t) = -0.81637\, t^{2.75}\, x^{2.5} + 1.12837\, t^{0.5}\, x $$

and

$$ u_{2,0}(x,t) = -0.50499\, t^{2.3} x^{3.3} + 2.25675\, t^{0.5}\, x $$

Then, the first and second approximate solutions of Eq. (25) which are denoted by $u_{1,1}(x,t)$ and $u_{1,2}(x,t)$, respectively while the first and second approximate solutions of Eq. (26) which are denoted by $u_{2,1}(x,t), u_{2,2}(x,t)$, are evaluated. The approximate solutions are computed for $x = 0.5, t \in [0,1], \Delta t = 0.1$. Also, comparison is then made with the exact solutions where the values are listed in Table V. From the results of Table V, the convergence and the accuracy of the obtained results between the exact and approximate solutions may be seen.

Also, Table VI, presents for comparison purpose the absolute errors between the approximate and the exact solutions for different values of x and t.

TABLE V. THE EXACT AND APPROXIMATE SOLUTIONS OF EXAMPLE 5.2

t	$u_1(x,t)$	$u_{1,1}(x,t)$	$u_{1,2}(x,t)$	$u_2(x,t)$	$u_{2,1}(x,t)$	$u_{2,2}(x,t)$
0	0	0	0	0	0	0
0.1	0.05	0.050074	0.0500001	0.1	0.100427	0.1000001
0.2	0.1	0.10046	0.100001	0.3	0.201474	0.200001
0.3	0.15	0.151286	0.150008	0.3	0.302937	0.300008
0.4	0.2	0.202589	0.200026	0.4	0.404692	0.400028
0.5	0.25	0.254334	0.250067	0.5	0.506674	0.500071
0.6	0.3	0.306423	0.300141	0.6	0.608861	0.60015
0.7	0.35	0.358698	0.350265	0.7	0.711279	0.70028
0.8	0.4	0.41094	0.400453	0.8	0.813993	0.800474
0.9	0.45	0.46287	0.450724	0.9	0.917109	0.900744
1	0.5	0.514143	0.501094	1	1.020776	1.001095

TABLE VI. THE ABSOLUTE ERRORS OF EXAMPLE 5.2.

t	$\lvert u_1(x,t)-u_{1,1}(x,t)\rvert$	$\lvert u_1(x,t)-u_{1,2}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,1}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,2}(x,t)\rvert$
0	0	0	0	0
0.1	7.417×10^{-5}	7.751×10^{-8}	4.266×10^{-4}	6.845×10^{-8}
0.2	4.597×10^{-4}	1.439×10^{-6}	1.474×10^{-3}	1.419×10^{-6}
0.3	1.286×10^{-3}	7.903×10^{-6}	2.937×10^{-3}	8.165×10^{-6}
0.4	2.589×10^{-3}	2.632×10^{-5}	4.692×10^{-3}	2.781×10^{-5}
0.5	4.334×10^{-3}	6.651×10^{-5}	6.674×10^{-3}	7.092×10^{-5}
0.6	6.423×10^{-3}	1.41×10^{-4}	8.861×10^{-3}	1.504×10^{-4}
0.7	8.689×10^{-3}	2.646×10^{-4}	0.011	2.804×10^{-4}
0.8	0.011	4.535×10^{-4}	0.014	4.744×10^{-4}
0.9	0.013	7.245×10^{-4}	0.017	7.439×10^{-4}
1	0.014	1.094×10^{-3}	0.021	1.095×10^{-3}

Similarly, if we choose other values for α_1 and α_2 such as $\alpha_1 = \alpha_2 = 1$, and substitute this value in Eq. (22), we can get the value of $g(x,t)$ and Eq. (22) will be:

$$ {}_0^C D_t^{\alpha_1} u(x,t) = x^2 - 0.64784\, t^2\, 2.33\, x^{3.6} + {}_a I_x^{\beta_1}\, {}_0 I_t^{\beta_1} \left[(xt)\big(u_1(x,t) + u_2(x,t)\big)\right] \tag{28} $$

and

$$ {}_0^C D_t^{\alpha_2} u_2(x,t) = 2x^2 - 0.26192\, t\, 1.45\, x^{3.8} + 0.27083\, t^2\, 2.45\, x^{2.8} + {}_a I_x^{\beta_2}\, {}_0 I_t^{\gamma_2} \left[(x^2 t) u_1(x,t)\right] \tag{29} $$

with the initial condition

$$ u_1(x,0) = u_2(x,0) = 0 \tag{30} $$

where, $\alpha_1, \alpha_2 = 1, \beta_1 = 0.5, \beta_2 = 0.25, \gamma_1 = 0.75,$ and $\gamma_2 = 0.3$ for all $(x,t) \in [0,1] \times [0,1]$

For comparison purpose, the exact solutions are given by $u_1(x,t) = xt$ and $u_2(x,t) = 2xt$. Now, by applying the VIM, and by using with the initial approximate solution as follows:

$$ u_{1,0}(x,t) = -0.64784\, t^{2.33}\, x^{3.6} $$

and

$$ u_{2,0}(x,t) = -0.26192\, t^{1.45}\, x^{3.8} + 0.27083\, t^{2.45}\, x^{2.8} $$

Then, the first and second approximate solutions of Eq. (28) which are denoted by $u_{1,1}(x,t)$ and $u_{1,2}(x,t)$, respectively while the first and second approximate solutions of Eq. (29) which are denoted by $u_{2,1}(x,t), u_{2,2}(x,t)$, are evaluated. The approximate solutions are computed for $x = 0.5, t \in [0,1], \Delta t = 0.1$. Also, comparison is then made with the exact solutions where the values are listed in Table 5.7. From the results of Table VII, the convergence and the accuracy of the obtained results between the exact and approximate solutions may be seen.

Also, Table VIII presents for comparison purpose the absolute errors between the approximate and the exact solutions for different values of x and t.

TABLE VII. THE EXACT AND APPROXIMATE SOLUTIONS OF EXAMPLE 5.2

t	$u_1(x,t)$	$u_{1,1}(x,t)$	$u_{1,2}(x,t)$	$u_2(x,t)$	$u_{2,1}(x,t)$	$u_{2,2}(x,t)$
0	0	0	0	0	0	0
0.1	0.05	0.049993	0.05	0.1	0.099991	0.1
0.2	0.1	0.099908	0.1	0.2	0.199914	0.2
0.3	0.15	0.149576	0.15	0.3	0.299671	0.3
0.4	0.2	0.198746	0.199999	0.4	0.399139	0.399998
0.5	0.25	0.24709	0.249995	0.5	0.498177	0.499992
0.6	0.3	0.294201	0.299984	0.6	0.59662	0.599975
0.7	0.35	0.339598	0.349958	0.7	0.694279	0.699937
0.8	0.4	0.382718	0.399902	0.8	0.790939	0.799858
0.9	0.45	0.422922	0.449792	0.9	0.886355	0.899708
1	0.5	0.459484	0.499594	1	0.98025	0.999445

TABLE VIII. THE ABSOLUTE ERRORS OF EXAMPLE 5.2.

t	$\lvert u_1(x,t)-u_{1,1}(x,t)\rvert$	$\lvert u_1(x,t)-u_{1,2}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,1}(x,t)\rvert$	$\lvert u_2(x,t)-u_{2,2}(x,t)\rvert$
0	0	0	0	0
0.1	6.853×10^{-6}	2.043×10^{-10}	8.639×10^{-6}	4.777×10^{-10}
0.2	9.246×10^{-5}	1.565×10^{-8}	8.562×10^{-5}	3.171×10^{-8}
0.3	4.245×10^{-4}	2.002×10^{-7}	3.293×10^{-4}	3.695×10^{-7}
0.4	1.254×10^{-3}	1.228×10^{-6}	8.609×10^{-4}	2.113×10^{-6}
0.5	2.91×10^{-3}	5.024×10^{-6}	1.823×10^{-3}	8.178×10^{-6}
0.6	5.799×10^{-3}	1.592×10^{-5}	3.38×10^{-3}	2.474×10^{-5}
0.7	0.01	4.224×10^{-5}	5.721×10^{-3}	6.314×10^{-5}
0.8	0.017	9.848×10^{-5}	9.061×10^{-3}	1.423×10^{-4}
0.9	0.027	2.079×10^{-4}	0.014	2.917×10^{-4}
1	0.041	4.06×10^{-4}	0.02	5.547×10^{-4}

6. CONCLUSIONS

In this article introduced efficacious technique for solving linear system of two dimensional FPIDEs utilizing the VIM. The VIM for solving linear system of two dimensional FPIDEs is formulated and the correction functional involved is determined. From there, convergence theorem of the sequence of approximate solution to the exact solution is provided and proved depending on the error function. The obtained results of the considered of the illustrative examples shows the reliability an applicability of the VIM for solving complicated differential equation.

Conflicts Of Interest

The authors declare that they have no competing interests.

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