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Research Article

Arcsine Ratio Sine Generalized Distributions with Applications to Biomedical and Engineering Data

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ABSTRACT

Trigonometric distributions offer a powerful approach to solving complex problems in probability and statistical modeling. In this study, we introduce a novel class of distributions: the Arcsine Ratio Sine Generalized (ARS-G) family. We provide a complete set of explicit formulas for the family's statistical properties. Our key illustration is the ARS-Weibull (ARS-W) distribution, which uses the Weibull model as its foundation. This particular model demonstrates remarkable flexibility, with its hazard function capable of assuming diverse shapes, including bump, bathtub, reversed bathtub, J-shaped, and L-shaped profiles. We thoroughly examine the ARS-W distribution's statistical properties and employ various established estimation methods to determine its parameters. Through rigorous Monte Carlo simulations, we confirm the consistency and stability of these estimation methods. We then showcase the ARS-W model's practical value by applying it to three real-world lifetime datasets: guinea pig survival times, active repair times for a communication transceiver, and turbocharger failure times. The model not only provides robust parameter estimates and a strong goodness-of-fit for the right-skewed guinea pig and transceiver data, but also outperforms traditional distributions (Weibull, Gumbel, Gamma, and Lognormal) for the left-skewed turbocharger dataset.

1. INTRODUCTION

The surge in distribution theory is driven by the restrictive nature of existing distributions in modeling complex datasets. Many classical distributions are limited, often containing only scale, shape, or location parameters. Others may be heavy-tailed, which complicates the control of skewness and kurtosis. Consequently, numerous studies have introduced greater flexibility and adaptability to these models by adding new parameters, modifying, extending, or generalizing existing distributions.

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Recent statistical literature highlights a growing trend of developing novel trigonometric distribution families. By integrating trigonometric functions into classical probability frameworks, these families provide enhanced flexibility for modeling lifetime data. This approach is evident in a wide range of contributions, including:

- **Sine-based families:** Sine-half-logistic inverse Rayleigh ([1]), Sine-G exponentiated exponential ([2]), Sine-G families ([3, 4]), Sin-G family in Bayesian survival modeling ([5]), a new class of the Sine-G family ([6]), exponent power Sine-G ([7]), novel Sine-G class ([8]), sine-Topp-Leone exponentiated-G ([9]), and Sine-π power odd-G ([10]).
- **Cosine-based families:** Cotangent exponentiated generalized generator ([11]), type-1 cosine exponentiated Weibull ([12]), Novel Arc Cosine Class ([13]), new class of cos-G family ([14]), new extended symmetric cosine ([15]), a new class of distributions using cosine and sine functions ([16]), extended Cos-G class ([17]), extended cosine generalized family ([18]), a cosine trigonometric distribution called the cosine-π power odd-G family ([19]), and the Cos-G class of Distributions ([20]).
- **Tangent-based families:** New logarithmic Tan-U family ([21]), Tan-G family ([22, 23]), and exponentiated cos and tan-generated distributions ([24]).
- **Other trigonometric distributions:** Arctan exponential ([25]), bounded lifetime distribution specified by a trigonometric function ([26]), arctan-Kumaraswamy exponential ([27]), new trigonometric-oriented distributional method ([28]), arcsine exponentiated-X family ([29]), hyperbolic Sine-G family ([30]), arctan-G family ([31]).

These numerous contributions have significantly advanced distribution theory and practice. The use of trigonometric functions has notably enhanced the flexibility of some baseline distributions, yielding interesting and practical results. The core motivation for the proposed Arcsine Ratio Sine (ARS) distribution is to overcome the limitations of existing distributions by offering enhanced flexibility for modeling complex datasets. While it's true that a multitude of distributions could be generated through structural alterations, the ARS family's significance lies in its practical value and concrete applications. It introduces a new parameter and integrates trigonometric functions, providing greater control over skewness and kurtosis, which is crucial for accurately modeling real-world data. For instance, in reliability engineering, the ARS-Weibull distribution can more precisely model the complex failure patterns of mechanical components—such as infant mortality followed by a wear-out phase—compared to a standard Weibull model, leading to more accurate predictions of product lifespan and improved maintenance schedules. Furthermore, its closed-form quantile function simplifies the generation of random numbers, making it highly valuable for Monte Carlo simulations in fields like finance, where it could be used to model and simulate the heavy-tailed, asymmetric behavior of asset returns more effectively than traditional distributions. This improved modeling capacity provides a more robust foundation for risk assessment and portfolio management.

This study is an ideal tool for facilitating digital security as it presents a new means of measuring the resiliency of Cyber-Physical Systems (CPS). Repair time data can seem like an ordinary measure of reliability, but our study redefines this physical performance metric to measure the vulnerability and resiliency of a system. We show how a statistical distribution of physical measurements can provide a standard of measure for a system's "mean time to recovery" from classes of digital and physical attacks. To accomplish this, we used statistical software to analyze physical data and created computer programs to simulate attack scenarios. These algorithms measure the recovery period of a system after attacks, enabling us to possess a new application of physical data for finding security vulnerabilities and estimating the resilience and recovery of a system from a cyber-physical attack. This new methodology and our findings bridge the physical system performance and cybersecurity gap by directly contributing to the new area of digital security. For more details, see [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48] .

The rest of this article is organized in this sequence; Section 2 contains the construction of the new family. The generic properties of the ARS family are studied in section 3. In Section 4, a special submodel is developed, called the ARS-Weibull distribution, with the estimation of its parameters done in Section 5. A simulation study is implemented in Section 6 to observe the behavior of the model parameters for various sample sizes. In Section 7, we demonstrate the applicability of the proposed NNS-w model using two real-life datasets, and the final comments are made in 8.

2. ARS GENERALIZED DISTRIBUTIONS

Let us consider the CDF of any baseline distribution represented by $G(x;\zeta)$ with the associated vector of parameters $\zeta > 0$. Using the T-X generator approach designed by [49], we propose a novel trigonometric probability distribution generator called the Arcsine ratio sine (ARS) generated family of distributions. Let the generalizer be the ratio of $\sin\left(\frac{\pi}{2}G(x;\zeta)\right)$, given as $W[G(x;\zeta)] = \frac{\theta(1-\sin(\frac{\pi}{2}G(x;\zeta))}{\theta+(2-\theta)\sin(\frac{\pi}{2}G(x;\zeta))}$, we utilize the transformer with probability density function (PDF) $r(t) = \cos(t)$, and for $\arcsin[W(G(x;\zeta))] < t < \frac{\pi}{2}$, the cumulative distribution function (CDF) of the ARS generalized family is

$$\begin{aligned} F(x;\theta,\zeta) &= -\int_{\arcsin[W(G(x;\zeta))]}^{\frac{\pi}{2}} \cos(t)dt \\ &= 1 - \frac{\theta(1-\sin\left(\frac{\pi}{2}G(x;\zeta)\right)}{\theta+(2-\theta)\sin\left(\frac{\pi}{2}G(x;\zeta)\right)}; \quad x>0, \quad \theta>0. \end{aligned} \tag{1}$$

The name "Arcsine ratio" stems from the fractional form of the generalizer, $W(G(x;\zeta))$. For $\theta = 2$, $F(x;\theta,\zeta) = 1 - \sin\left(\frac{\pi}{2}G(x;\zeta)\right)$, which is the CDF of sine-G family of distributions. Differentiating the CDF of the ARS family of distributions defined in Eq. (1), the corresponding PDF is obtained thus;

$$f(x;\theta,\zeta) = \frac{\pi\theta\cos\left(\frac{\pi}{2}G(x;\zeta)\right)g(x;\zeta)}{\left(\theta+(2-\theta)\sin\left(\frac{\pi}{2}G(x;\zeta)\right)\right)^2}, \quad \text{where,} \quad x,\theta>0. \tag{2}$$

From the CDF and PDF of the ARS family of distributions defined in Eqs. (1) and (2), the survival function $S(x;\zeta)$ and hazard function $h(x;\zeta)$ of the ARS-G family of distribution can be defined as

$$S(x;\theta,\zeta) = \frac{\theta(1-\sin\left(\frac{\pi}{2}G(x;\zeta)\right)}{\theta+(2-\theta)\sin\left(\frac{\pi}{2}G(x;\zeta)\right)},$$

and

$$h(x;\theta,\zeta) = \frac{\pi\cos\left(\frac{\pi}{2}G(x;\zeta)\right)g(x;\zeta)}{[1-\sin\left(\frac{\pi}{2}G(x;\zeta)\right)][\theta+(2-\theta)\sin\left(\frac{\pi}{2}G(x;\zeta)\right)]}, \tag{3}$$

respectively. The curvatures of Eqs.(2) and (3) are very useful for defining a suitable statistical model to fit given lifetime data sets.
The quantile function of the ARS family of distribution can be obtained by solving the equation $x_{(q;\zeta)} = F^{-1}(q;\zeta)$. Following this, therefore, the quantile function of the ARS-G distributions is provided as

$$x_{(q;\zeta)} = G^{-1}\left(\frac{2}{\pi}\arcsin\left(\frac{\theta q}{2-q(2-\theta)}\right);\zeta\right); \quad q \in U(0,1).$$

The random deviate generation can be done using

$$x_{(q;\zeta)} = G^{-1}\left(\frac{2}{\pi}\arcsin\left(\frac{\theta q}{2-q(2-\theta)}\right);\zeta\right),$$

where $q \in U(0,1)$. In addition to using the quantile function to generate random numbers, $Q(q;\zeta)$ is useful when studying the properties of the ARS family of distributions.

Theorem 1 (Identifiability)**.** *A parametric family of probability distributions $\{F(x;\boldsymbol{\theta},\boldsymbol{\zeta}) : \boldsymbol{\theta} \in \Theta, \boldsymbol{\zeta} \in Z\}$ is said to be* ***identifiable*** *if for any two parameter vectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \Theta$ and baseline parameters $\boldsymbol{\zeta} \in Z$, the equality of the cumulative distribution functions $F(x;\boldsymbol{\theta}_1,\boldsymbol{\zeta}) = F(x;\boldsymbol{\theta}_2,\boldsymbol{\zeta})$ for all x in the support of the distribution implies that $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$.*
More specifically for the ARS distribution where $\boldsymbol{\theta}$ is a scalar shape parameter θ and $\boldsymbol{\zeta}$ are baseline parameters, the parameter θ is identifiable if $F(x;\theta_1,\boldsymbol{\zeta}) = F(x;\theta_2,\boldsymbol{\zeta})$ for all x implies $\theta_1 = \theta_2$.

Proof of Identifiability. Let's assume that for two parameter values, θ_1 and θ_2, the CDFs are identical for all x in the domain of the distribution. That is, $F(x,\theta_1,\zeta) = F(x,\theta_2,\zeta)$ for all $x \in \mathbb{R}$.
Let $S(x;\zeta) = \sin\left(\frac{\pi}{2}G(x;\zeta)\right)$. As x varies over the support of the baseline distribution, $G(x;\zeta)$ ranges from 0 to 1, and thus $\frac{\pi}{2}G(x;\zeta)$ ranges from 0 to $\frac{\pi}{2}$. Consequently, $S(x;\zeta)$ can take on a continuum of values within [0, 1].
From the definition of the ARS CDF, we have:

$$1 - \frac{\theta_1(1-S(x;\zeta))}{\theta_1+(2-\theta_1)S(x;\zeta)} = 1 - \frac{\theta_2(1-S(x;\zeta))}{\theta_2+(2-\theta_2)S(x;\zeta)}$$

Subtracting 1 from both sides and multiplying by -1, we get:

$$\frac{\theta_1(1 - S(x;\zeta))}{\theta_1 + (2 - \theta_1)S(x;\zeta)} = \frac{\theta_2(1 - S(x;\zeta))}{\theta_2 + (2 - \theta_2)S(x;\zeta)}$$

Let $S = S(x;\zeta)$ for brevity.

$$\frac{\theta_1(1 - S)}{\theta_1 + (2 - \theta_1)S} = \frac{\theta_2(1 - S)}{\theta_2 + (2 - \theta_2)S}$$

We need to consider two cases for $(1 - S)$:

1. If $1 - S \neq 0$: (i.e., $S \neq 1$, which means $G(x;\zeta) \neq 1$) In this case, we can divide both sides by $(1 - S)$:

$$\frac{\theta_1}{\theta_1 + (2 - \theta_1)S} = \frac{\theta_2}{\theta_2 + (2 - \theta_2)S}$$

   Cross-multiplying gives:

$$\theta_1(\theta_2 + (2 - \theta_2)S) = \theta_2(\theta_1 + (2 - \theta_1)S)$$

$$\theta_1\theta_2 + \theta_1(2 - \theta_2)S = \theta_2\theta_1 + \theta_2(2 - \theta_1)S$$

   Subtracting $\theta_1\theta_2$ from both sides:

$$\theta_1(2 - \theta_2)S = \theta_2(2 - \theta_1)S$$

   Since this equality must hold for all x (meaning S can take on a continuum of values in $[0, 1)$), we can choose an x such that $S \neq 0$ (e.g., any x for which $G(x;\zeta) > 0$). Assuming $S \neq 0$:

$$\theta_1(2 - \theta_2) = \theta_2(2 - \theta_1)$$

$$2\theta_1 - \theta_1\theta_2 = 2\theta_2 - \theta_2\theta_1$$

   Adding $\theta_1\theta_2$ to both sides:

$$2\theta_1 = 2\theta_2$$

$$\theta_1 = \theta_2$$

2. If $1 - S = 0$: (i.e., $S = 1$, which means $G(x;\zeta) = 1$) In this case, the original equation becomes $F(x, \theta_1, \zeta) = 1$ and $F(x, \theta_2, \zeta) = 1$. This means both CDFs are 1 at values of x where the baseline CDF reaches 1. This condition is satisfied for any θ_1, θ_2 and does not provide information to distinguish them. However, since the previous case ($S \neq 1$) covers almost all x in the domain (except potentially a single point at the end of the support where $G(x;\zeta) = 1$), and the identity $\theta_1 = \theta_2$ was derived for that broad range, the identifiability holds.

   Specifically, the function $H(S) = \frac{\theta(1-S)}{\theta+(2-\theta)S}$ must be the same for all $S \in [0, 1]$. We have shown that if $H(S)$ is the same for $S \in [0, 1)$, then $\theta_1 = \theta_2$. The fact that $H(1) = 0$ for any θ is consistent with this.

Since the equality $F(x, \theta_1, \zeta) = F(x, \theta_2, \zeta)$ must hold for all x in the support of the random variable, and this implies that the function $\frac{\theta(1-S)}{\theta+(2-\theta)S}$ must be identical for all valid values of S, including a continuum of values where $S \neq 0$ and $S \neq 1$. The derived algebraic manipulation demonstrates that this is only possible if $\theta_1 = \theta_2$.
Therefore, the ARS distribution family is identifiable with respect to its shape parameter θ. This completes the proof.

2.1 Expansion of the ARS family

Let $G(x;\zeta)$ be the CDF of the baseline random variable and $g(x;\zeta)$ be its PDF, then the CDF in Eq. (1) and the PDF in Eq. (2) can be represented in linear form to enhance analytical flexibility, (provided $\theta > 1$) using the following existing series expansion and trigonometric representations.

$$\begin{gathered} (1+y)^{-n} = \sum_{i=0}^{\infty}(-1)^i\binom{n+i-1}{i}y^i; \quad (1+x)^{-2} = \sum_{h=1}^{\infty}(-1)^{h-1}hx^{h-1}, \\ \cos(x) = \sum_{j=0}^{\infty}(-1)^j\frac{(x)^{2j}}{(2j)!}; \quad \sin(x) = \sum_{k=0}^{\infty}(-1)^k\frac{(x)^{2k+1}}{(2k+1)!}. \end{gathered} \tag{4}$$

Here we utilize another important existing result on power series raised to a positive integer established in [50] and used by [24].

$$\left(\sum_{v=0}^{\infty} b_v y^v\right)^s = \sum_{v=0}^{\infty} d_{v,m} y^v \tag{5}$$

where $d_{v,m} = (vb_0)^{-1} \sum_{p=1}^{v} [m(p-1)-v] b_p d_{v-p}, v-m$ and $d_{0,m} = b_0^m$.

Proof. The CDF and PDF provided in Eqs. (1) and (2) can be expressed as a linear combination of the powers of CDFs and PDFs.

$$F(x;\zeta) = \sum_{i,j,k}^{\infty} \sum_{l=0}^{\infty} w_i d_{j+k,q} G^{2l+j+k}(x;\zeta)$$

where $w_i = (-1)^{i+j+k} \binom{s}{i}\binom{i}{k}\binom{i+j-1}{j} (\frac{\pi}{2})^{j+k}$ and

$$f(x;\theta,\zeta) = \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \omega_i d_{k,m} g(x;\zeta)(G^{2(j+k)+i-1}(x;\zeta))$$

Proof. Using the series in Eq. (4), the PDF of the ARS family of distributions in Eq. (2) becomes

$$f(x;\theta,\zeta) = \sum_{j=0}^{\infty} \omega_i g(x;\zeta)(G(x;\zeta))^{2j} \left(\sum_{k=0}^{\infty} \frac{(-1)^k (\frac{\pi}{2})^{2k+1} (G(x;\zeta))^{2k+1}}{(2k+1)!}\right)^{i-1}$$

where $\omega_i = \sum_{i=1}^{\infty} \frac{(-1)^{i+j-1} i(2-\theta)^{i-1} i\pi^{2j+1}}{\theta^i (2j)! 2^{2j}}$
Applying the result in Eq. (5)

$$\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} (G(x;\zeta))^{2k+1}\right)^{i-1} = (G(x;\zeta))^{i-1} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} (G^2(x;\zeta))^k\right)^{i-1}$$

$$= (G(x))^{i-1} \left(\sum_{k=0}^{\infty} b_k (G^2(x))^k\right)^{i-1} = \sum_{k=0}^{\infty} d_{k,m} (G(x;\zeta))^{2k+i-1} \tag{6}$$

where $d_{k,m} = \frac{(-1)^k}{(2k+1)!} (\frac{\pi}{2})^{2k+1}$ and $d_{k,m} = (kb_0)^{-1} \sum_{p=1}^{k} [m(p-1)-k] b_p d_{k-p}, k-m$ and $d_{0,m} = b_0^m$. Then, by putting Eq. (5) into (6), we have

$$f(x;\theta,\zeta) = \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \omega_i d_{k,m} g(x;\zeta)(G(x;\zeta))^{2(j+k)+i-1}$$

By integration under the sums of the CDF of the ARS-G distributions as

$$F(x) = \sum_{i,j,k}^{\infty} \sum_{l=0}^{\infty} w_i d_{l,m} (G(x))^{2l+j+k}; \quad \text{where} \quad d_{l,m} = \frac{(-1)^l}{(2l+1)!} (\frac{\pi}{2})^{2l+1}.$$

Detail proof is contained in Appendix B

3. GENERIC PROPERTIES OF ARS FAMILY OF DISTRIBUTIONS

In this section, we consider some generic properties of the proposed ARS family of distributions. These include moment, incomplete moment, order statistic, entropy, and score function.

3.1 Moments

In statistics and distribution theory, the moments of the random variable X play a very significant role in describing the properties of a distribution. Generally, the non-central moment of k^{th} order of a distribution is defined as

$$\mu_k^{'} = \int_{-\infty}^{\infty} x^k f(x;\zeta)dx$$

Using the PDF in Eq. (4) above, the k^{th} non-central moment of the ARS family of distribution is

$$\mu_k^{'} = \sum_{j=0}^{\infty}\sum_{p=0}^{\infty} \omega_i d_{2k+1,i-1} \int_{-\infty}^{\infty} x^k g(x;\zeta)(G(x;\zeta))^{2(j+k)+i-1} dx$$

From $\mu_k^{'}$ one can derive moment-based measures such as mean, variance, skewness, kurtosis, and the Coefficient of variation. Using the same approach, the expansion of the central moment of order r is obtained as follows. Let $\mu_r^{'}$ be the central moment of order r of the ARS-G class and $\mu^{2(j+k)+i-1}$ represent the moment of order r of the exponentiated-G class, with parameter $2(j+k)+i-1$. The

$$\mu_r^{'} = \int_{-\infty}^{\infty} (x-\mu)^r dF(x)$$

Applying the binomial theory, we obtain

$$\mu_r^{'} = \sum_{m=0}^{r}(-1)^m \binom{r}{m} \mu^m \int_{-\infty}^{\infty} x^{r-m} dF(x) = \sum_{m=0}^{r}(-1)^m \binom{r}{m} \mu^m \mu_{r-m}$$

$$= \sum_{m=0}^{r}(-1)^m \binom{r}{m} \mu^m \sum_{j=0}^{\infty}\sum_{p=0}^{\infty} \omega_i d_{2k+1,i-1} \int_{-\infty}^{\infty} x^k g(x;\zeta)(G(x;\zeta))^{2(j+k)+i-1} dx$$

Furthermore, the probability-weighted moment is defined by

$$W_{k,s}(x;\theta,\zeta) = E(x^k F^s(x;\zeta) = \int_{-\infty}^{\infty} x^k F^s(x;\zeta) f(x;\zeta)dx$$

Let $F(x;\zeta)$ and $f(x;\zeta)$ be the CDF and PDF given in Eqs. (3) and (4) respectively, the probability-weighted moment of the ARS family of distributions is

$$W_{k,s}(x;\zeta) = \sum_{l=0}^{\infty} T_{i,j,k} \int_{-\infty}^{\infty} x^k g(x;\zeta) G^{2k}(x;\zeta)(\sin\left(\frac{\pi}{2}G(x;\zeta)\right))^{j+l} dx$$

where $T_{i,j,k} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j+k+l}\binom{s}{i}\binom{i}{j}\binom{i+l+1}{l}\frac{\pi^{2k+1}(2-\theta)^l}{(2k)!2^{2k}\theta^{i+1}}$.

In addition, the moment-generating function (MGF) is defined as

$$M_{X;\zeta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!}\mu_r^{'}$$

The final form is

$$= \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^m w_i d_{2k+1,i-1}}{m!} \int_{-\infty}^{\infty} x^k g(x;\zeta)(G(x;\zeta))^{2(j+k)+i-1} dx.$$

3.2 Incomplete Moment

The *rth* incomplete of a random variable X which assumes the ARS family of distributions can be expressed as follows;

$$\mathrm{IMC}_r(q) = \int_0^q x^r f(x;\zeta) = \int_0^q x^r \frac{\pi\theta\cos\left(\frac{\pi}{2}G(x;\zeta)\right) g(x;\zeta)}{[\theta+(2-\theta)\sin\left(\frac{\pi}{2}G(x;\zeta)\right)]^2}$$

After some algebraic manipulations, we obtain the incomplete moment of the ARS family of distributions as

$$= \sum_{j=0}^{\infty}\sum_{p=0}^{\infty} \omega_i d_{2k+1,i-1} \int_0^q x^r g(x;\zeta)(G(x;\zeta))^{2(j+k)+i-1} dx$$

3.3 Order Statistics

Order statistics is very crucial in several fields of statistical theory and methods, especially when the interest is in survival analysis studies and quality control processes. The order statistics of the ARS family of distributions are presented here. Let $X_1, X_2, \cdots, X_n$ be random samples selected from the ARS family of distributions and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ represent the order statistics, then the probability density function of the i^{th} order statistics $X_{i:n}$ for $1 \leq i \leq n$ is

$$f_{i:n}(x) = B_{i:n} f(x;\zeta) F^{i-1}(x;\zeta)(1 - F(x;\zeta))^{n-i}; x \in \Re, \tag{7}$$

where

$$B_{i:n} = \frac{n!}{(i-1)!(n-1)!}.$$

Substituting Eq. (3) and (4) into Eq. (7) and performing expansions, the above density can be written as

$$= B_{i,n} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{\pi\theta \cos\left(\frac{\pi}{2} G(x;\zeta)\right) g(x;\zeta)}{[\theta + (2-\theta) sin(\frac{\pi}{2} G(x;\zeta))]^2} \left[\frac{\theta(1 - \sin\left(\frac{\pi}{2} G(x;\zeta)\right))}{\theta + (2-\theta)\sin\left(\frac{\pi}{2} G(x;\zeta)\right)} \right]^{j+i-1}$$

Applying a binomial series expansion, we obtain

$$= B_{i:n} \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{n-i}{j} \binom{j+k-1}{k} \theta^{j+k} \pi \frac{g(x;\zeta)\cos(\frac{\pi}{2} G(x;\zeta))(sin(\frac{\pi}{2} G(x;\zeta)))^k}{[\theta(1 + \frac{(2-\theta) sin(\frac{\pi}{2} G(x;\zeta))}{\theta})]^{j+l-1}}$$

$$= B_{i:n} \sum_{j=0}^{n-i} \sum_{k}^{\infty} \sum_{l}^{\infty} (-1)^{j+k+l} \binom{n-i}{j} \binom{j+k-1}{k,l} \theta^{k+l-i+1} \pi (2-\theta)^l g(x;\zeta)$$
$$\cos\left(\frac{\pi}{2} G(x;\zeta)\right) (\sin\left(\frac{\pi}{2} G(x;\zeta)\right))^{k+l}$$

$$= B_{i:n} \sum_{m=0}^{\infty} Z_{j,k,l} g(x;\zeta)(G(x;\zeta))^{2m} (sin(\frac{\pi}{2} G(x;\zeta)))^{k+l}$$

$$= B_{i:n} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} Z_{j,k,l} b_r g(x;\zeta)(G(x))^{2(m+r)+k+l}$$

where

$$Z_{j,k,l} = \frac{(-1)^{j+k+l}\binom{n-i}{j}\binom{j+k-1}{k,l}\theta^{k+l-i+1}\pi^{m+1}(2-\theta)^l}{2^m(2m)!}$$

Then, PDF of the i^{th} smallest order statistic is obtained when $i = 1$ in

$$f_{1:n}(x) = n \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} Z_{j,k,l} b_r g(x;\zeta)(G(x))^{2(m+r)+k+l}$$

Similarly, the PDF of the i^{th} largest order statistics is obtained when $i = n$ in

$$f_{n:n}(x) = \mathbf{n} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} Z'_{0,k,l} b_r g(x;\zeta)(G(x))^{2(m+r)+k+l},$$

where $Z'_{0,k,l}$ is the coefficient $Z_{j,k,l}$ evaluated at $i = n$ and $j = 0$.

3.4 Entropy

Rényi Entropy is an extension of Shannon entropy, it has several applications in information and application theory, economics, physics, and other complex schemes. The certainty of observing an event with high entropy is very low compared to a system with lower entropy. Here, the Rényi entropy measure using information associated with random variable X is defined as

$$I_R(s) = \frac{1}{1-s} \log\left(\int_0^{\infty} f^s(x)dx\right), s \neq 1, s > 0 \tag{8}$$

From the PDF of ARS-G, $f^s(x)$ can be written as

$$f^s(x) = \frac{(\pi\theta)^s \cos^s(\frac{\pi}{2}G(x;\zeta))g^s(x;\zeta)}{[\theta + (2-\theta)\sin(\frac{\pi}{2}G(x;\zeta))]^{2s}}$$

Let us consider the application of the Taylor series expansion of the trigonometric function

$$(\cos(\frac{\pi}{2}G(x;\zeta)))^s = \left(\sum_{k=0}^{\infty}(-1)^k\left(\frac{\pi}{2}\right)^{2k}\frac{((G^2(x;\zeta)))^k}{(2k)!}\right)^s$$

and also the series expansion of $(1+x)^{-2} = \sum_{k=1}^{\infty}(-1)^{k-1}kx^{k-1}$, then

$$\theta^{-2s}\left(1+\left(\frac{2-\theta}{\theta}\right)(\sin(\frac{\pi}{2}G(x;\zeta)))\right)^{-2s} = \sum_{v=1}^{\infty}(-1)^{v-1}v\frac{(2-\theta)^{v-1}}{\theta^{v+2s-1}}(\sin(\frac{\pi}{2})G(x;\zeta))^{v-1} \tag{9}$$

Applying series expansion in Eq. (9), we have

$$= \sum_{v=1}^{\infty}(-1)^{v-1}v\frac{(2-\theta)^{v-1}}{\theta^{v+2s-1}}\left(\sum_{w=0}^{\infty}(-1)^w(\frac{\pi}{2})^{2w+1}\frac{(G(x;\zeta))^{2w+1}}{(2w+1)!}\right)^{v-1}$$

$$= \sum_{v=1}^{\infty}(-1)^{v-1}v\frac{(2-\theta)^{v-1}}{\theta^{v+2s-1}}(G(x;\zeta))^{v-1}\left(\sum_{w=0}^{\infty}b_w(G(x;\zeta))^{2w}\right)^{v-1}$$

Using Eq. (8), we obtain

$$I_R(s) = \frac{1}{1-s}\log\left(\sum_{v=1}^{\infty}\sum_{w=0}^{\infty}t_v b_w\int_0^{\infty}(G(x;\zeta))^{v-1}((G(x;\zeta))^2)^w g^s(x;\zeta)dx\right)$$

where $t_v = (-1)^{v-1}\frac{v\pi^s(2-\theta)^{v-1}}{\theta^{v+2s-1}}$, $b_w = \frac{(-1)^w(\frac{\pi}{2})^{2w+1}}{(2w+1)!}$ and $e_0 = b_0^{v-1}, e_m = \frac{1}{mb_0}\sum_{w=1}^{m}(w(v-1)-m+w)b_w e_{m-w}; m \geq 1$

$$= \frac{1}{1-s}\log\left(\sum_{v=1}^{\infty}\sum_{w=0}^{\infty}t_v d_w\int_0^{\infty}(G(x;\zeta))^{2w+v-1}g^s(x;\zeta)dx\right)$$

3.5 Score Function of ARS family

Given that $\underline{x} = (x_1, \cdots, x_n)^T$ represents a random sample of sin n chosen from the ARS-G with an unknown vector of parameters $\underline{w}$. The log-likelihood (LL) function for w is

$$\begin{aligned} \mathfrak{l}(\underline{w}) = {} & n\log(\pi\theta) + \sum_{i=1}^{n}\log\left(\cos\left(\frac{\pi}{2}G(x_i|\underline{w})\right)\right) + \sum_{i=1}^{n}\log\left(g(x_i|\underline{w})\right) \\ & - 2\sum_{i=1}^{n}\log\left(\theta + (2-\theta)\sin\left(\frac{\pi}{2}G(x_i|\underline{w})\right)\right), \end{aligned}$$

then, the score function for the ARS family of distributions is obtained as

$$\begin{aligned} U(w_j) = {} & \sum_{i=0}^{n}\frac{1}{g(x_i|\underline{w})}\frac{\partial g(x_i|\underline{w})}{\partial w_j} - \frac{\pi}{2}\sum_{i=1}^{n}tan(\frac{\pi}{2}G(x_i|\underline{w})\frac{\partial G(x_i|\underline{w})}{\partial w_j} \\ & - \pi(2-\theta)\sum_{i=1}^{n}\frac{\cos(\frac{\pi}{2}G(x_i|\underline{w}))}{\theta + (2-\theta)sin(\frac{\pi}{2}G(x_i|\underline{w}))}\frac{\partial G(x_i|\underline{w})}{\partial w_j}. \end{aligned}$$

4. SUB-MODEL: THE ARS-WEIBULL DISTRIBUTION

This section focuses on a special distribution of the ARS family named the ARS-Weibull (ARS-W) distribution. The baseline distribution is one of the most popular distributions used in survival and reliability analysis, called the Weibull distribution proposed by [51]. In addition, several Weibull-based probability distributions have been implemented in the literature because they provide a good fit for numerous forms of data (see [52, 53, 54, 55, 56, 57], among others). However, the Weibull distribution does not offer a nonmonotonic failure rate, which is common in reliability and survival studies. Its implementation in this study as a baseline distribution will enhance the flexibility of the properties and improve the analytical tractability and computational ability. The CDF and PDF of the baseline distribution are respectively

$$G(x;\lambda,\omega) = 1 - e^{-\lambda x^{\omega}}; \quad x > 0, \quad \lambda > 0, \quad \omega > 0, \tag{10}$$

and

$$g(x;\lambda,\omega) = \lambda\omega x^{\omega-1} e^{-\lambda x^{\omega}}; \quad x > 0, \quad \lambda > 0, \quad \omega > 0. \tag{11}$$

The ARS-G trigonometric transform class is applied to the Weibull distribution. By substituting Eq. (10) into (1), we obtain the CDF of the ARS-W distribution as

$$F(x;\theta,\lambda,\omega) = 1 - \left(\frac{\theta\left(1 - \sin\left(\frac{\pi}{2}(1 - e^{-\lambda x^{\omega}}\right)\right)}{\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x^{\omega}})\right)} \right). \tag{12}$$

Similarly, substituting Eqs. (10) and (11) into Eq. (2), we obtain the corresponding PDF of the ARS-W model as follows;

$$f(x;\theta,\lambda,\omega) = \frac{\pi\theta\lambda\omega\cos\left(\frac{\pi}{2}(1 - e^{-\lambda x^{\omega}})\right) x^{\omega-1} e^{-\lambda x^{\omega}}}{(\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x^{\omega}})\right))^2}, \tag{13}$$

where $x > 0, \theta > 0, \lambda > 0$ and $\omega > 0$.

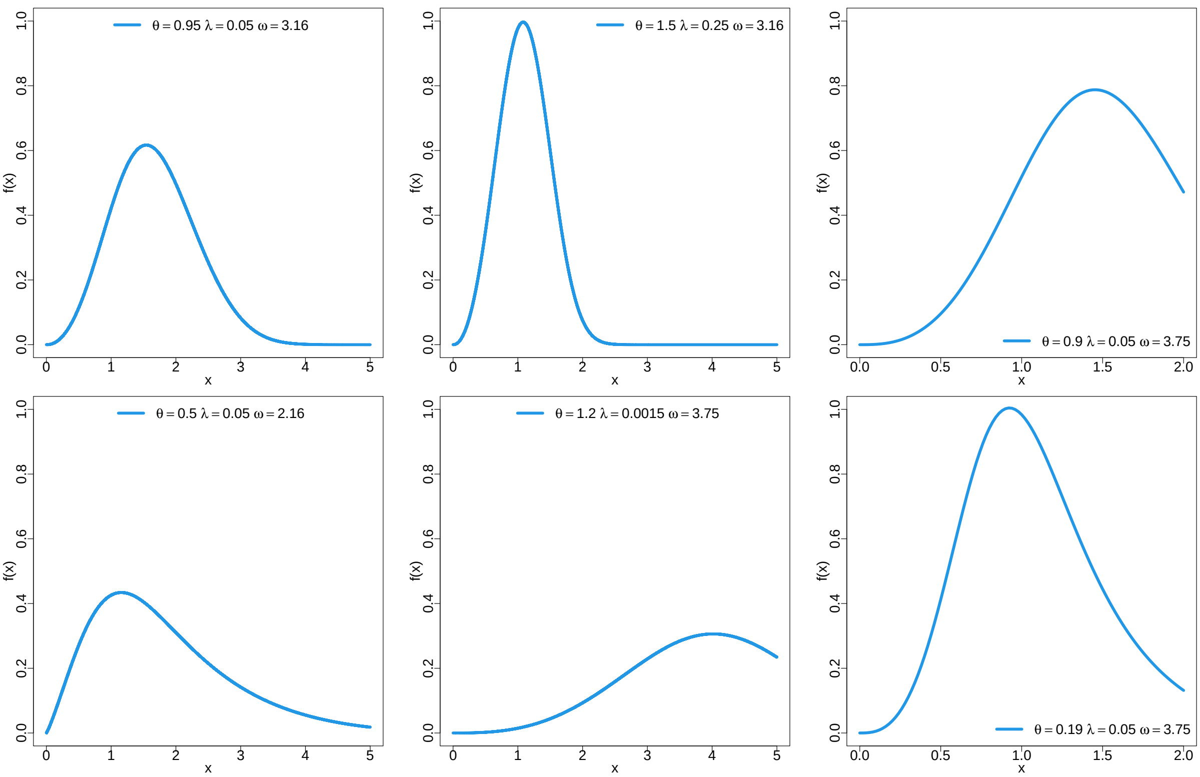

Fig. 1. PDF of ARS-W$(\theta, \lambda, \omega)$

Figure (1) shows the PDF plots of the ARS-W distribution for some parameter values chosen. The ARS-W distribution displays robust and flexible features such as unimodal shape, high-peak, low-peak, right-skewed, left-skewed, and almost symmetrical curves with unimodal densities. The graph also shows exponential curves. The ARS-W distribution is more flexible than the reference distribution. The survival function $S(x;\theta,\lambda,\omega)$ and hazard function $h(x;\theta,\lambda,\omega)$ of the ARS-W distribution take the forms

$$S(x;\theta,\lambda,\omega) = \frac{\theta(1 - \sin(\frac{\pi}{2}(1 - e^{-\lambda x^{\omega}}))}{\theta + (2-\theta)\sin(\frac{\pi}{2}(1 - e^{-\lambda x^{\omega}})},$$

and

$$h(x;\theta,\lambda,\omega)=\frac{\pi\lambda\omega\cos(\frac{\pi}{2}(1-e^{-\lambda x^{\omega}}))x^{\omega-1}e^{-\lambda x^{\omega}}}{[1-\sin(\frac{\pi}{2}(1-e^{-\lambda x^{\omega}}))][\theta+(2-\theta)\sin(\frac{\pi}{2}(1-e^{-\lambda x^{\omega}}))]},$$

respectively.

Figure (2) represents the graphs of the hazard function for various parameter values of the ARS-W distribution. From the plots, hazards depicting bathtub-shaped or reversed bathtub-shaped, bumped-shaped, bump-shaped, increasing, and decreasing behaviours can be modeled by the proposed ARS-W distribution.

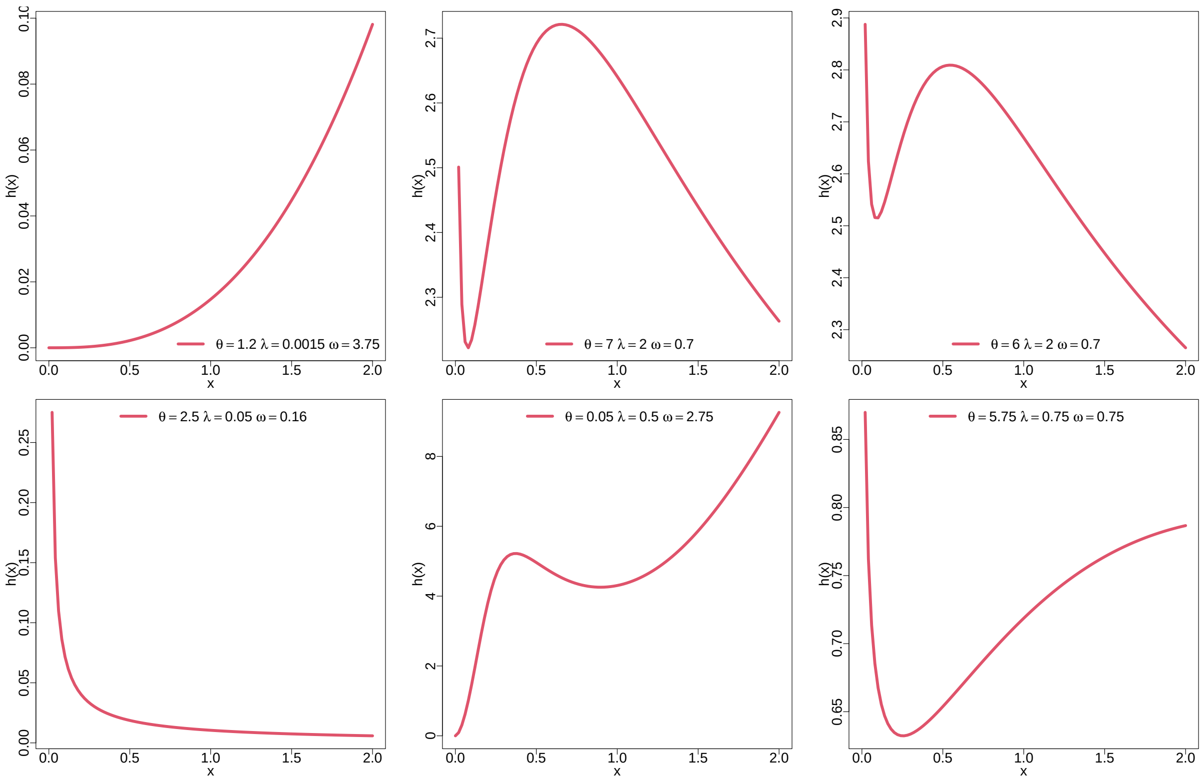

Fig. 2. Hazard function of ARS-W(θ,λ,ω)

4.1 Special Cases of the ARS-W distribution

The ARS-W distribution contains some special models:

i When $\omega=1$, the ARS-W model reduces to the ARS-Rayleigh model.

ii When $\omega=2$, the ARS-W model reduces to the ARS-Exponential model.

All the results for the ARS family above could be applied to the ARS-W distribution. Specifically, the quantile function of the ARS-W distribution is presented as

$$Q(u)=\left[-\frac{1}{\lambda}\log\left(1-\frac{2}{\pi}\arcsin\left(\frac{\theta u}{2-(2-\theta)u}\right)\right)\right]^{\frac{1}{\omega}};\quad 0\le u\le 1 \tag{14}$$

The second quartile of the ARS-W distribution can be evaluated by equating $u=0.5$, which is

$$\text{median}=\left[-\frac{1}{\lambda}\log\left(1-\frac{2}{\pi}\arcsin\left(\frac{\theta}{2-\theta}\right)\right)\right]^{\frac{1}{\omega}}$$

Similarly, the quartiles, percentiles and deciles can be calculated by substituting the appropriate values into Eq. (14). The quantile function is widely used in risk management studies, fields that require percentile-based decision making (e.g., in finance or engineering), and random number generation.

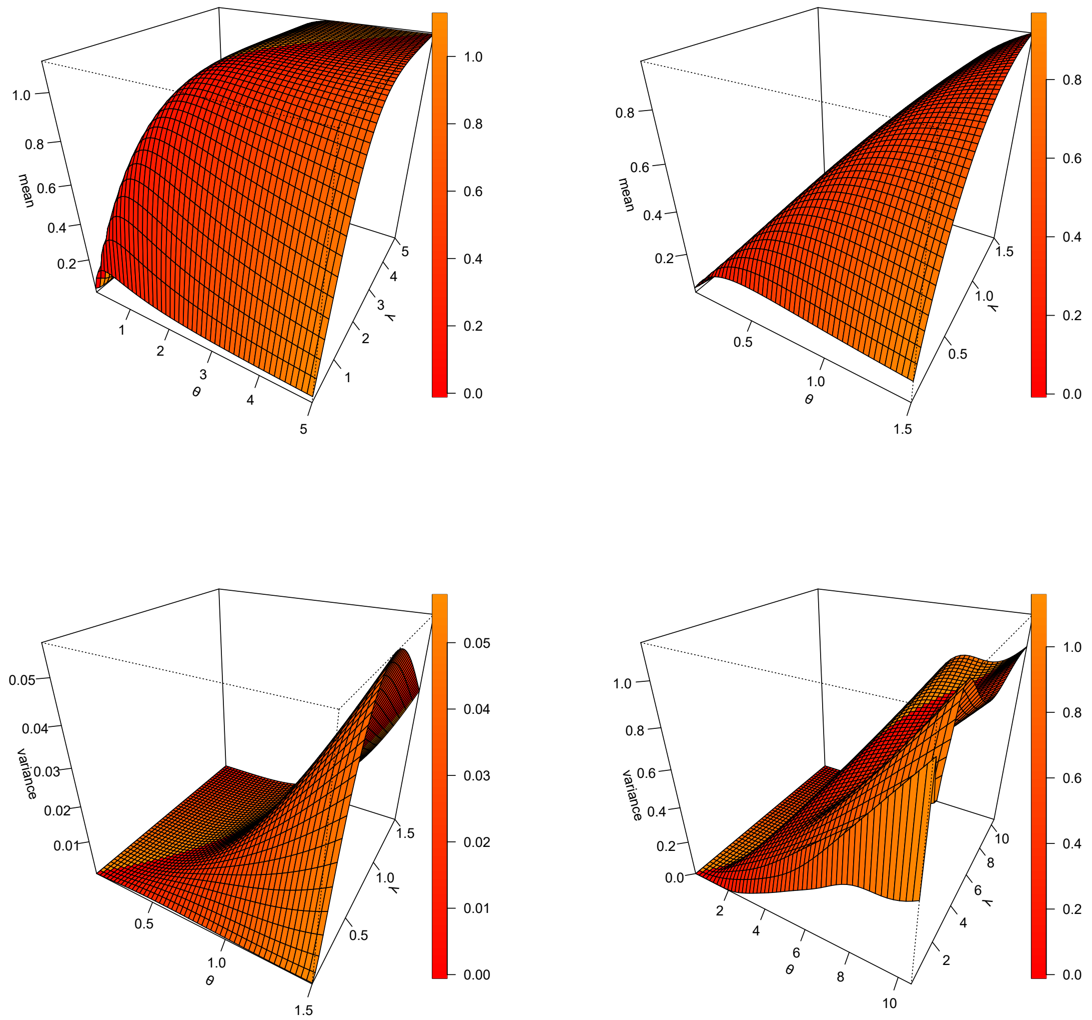

Fig. 3. Mean and Variance of ARS-W$(\theta, \lambda, \omega)$

Figure (3) illustrates the behavior of the mean and variance of the ARS-W$(\theta, \lambda, \omega)$ distribution as its parameters vary. The four 3D surface plots exhibit various trends for central tendency and dispersion. The upper plots, which are for the mean, exhibit a monotonically increasing trend. In the upper-left plot (parameters 1-5), the mean rises sharply and then stabilizes at about 1.0, suggesting a saturation point. The upper-right plot (parameters 0.5-1.5) also shows an increasing mean, though with a different sensitivity, up to about 0.8. These are suggestive of a positive correlation between the mean and the varying parameters. The lower plots are for the variance, and they show more complicated behaviors. The lower-left plot (parameters 0.5-1.5) shows the variance generally low (0.01-0.02) but increasing steeply as both parameters approach their upper limit (around 1.5), up to 0.05. This implies sensitivity to larger parameter values. Most noticeably, the lower-right panel (parameters 2-10) reveals a clear "U-shaped" or "bowl-shaped" pattern, with variance smallest in some intermediate range (e.g., parameters 4-6). Departure from this range, either downward or upward, leads to wildly larger variance, as much as 1.0. This key finding implies that some parameter choices are required to maintain the variability of the distribution at a minimum.

In effect, while the mean of the ARS-W distribution increases with its parameters, its variance has a more complex dependence, where some intervals of parameters lead to minimized dispersion. This indicates the need for parameter optimization in order to control the stability and spread of the distribution.

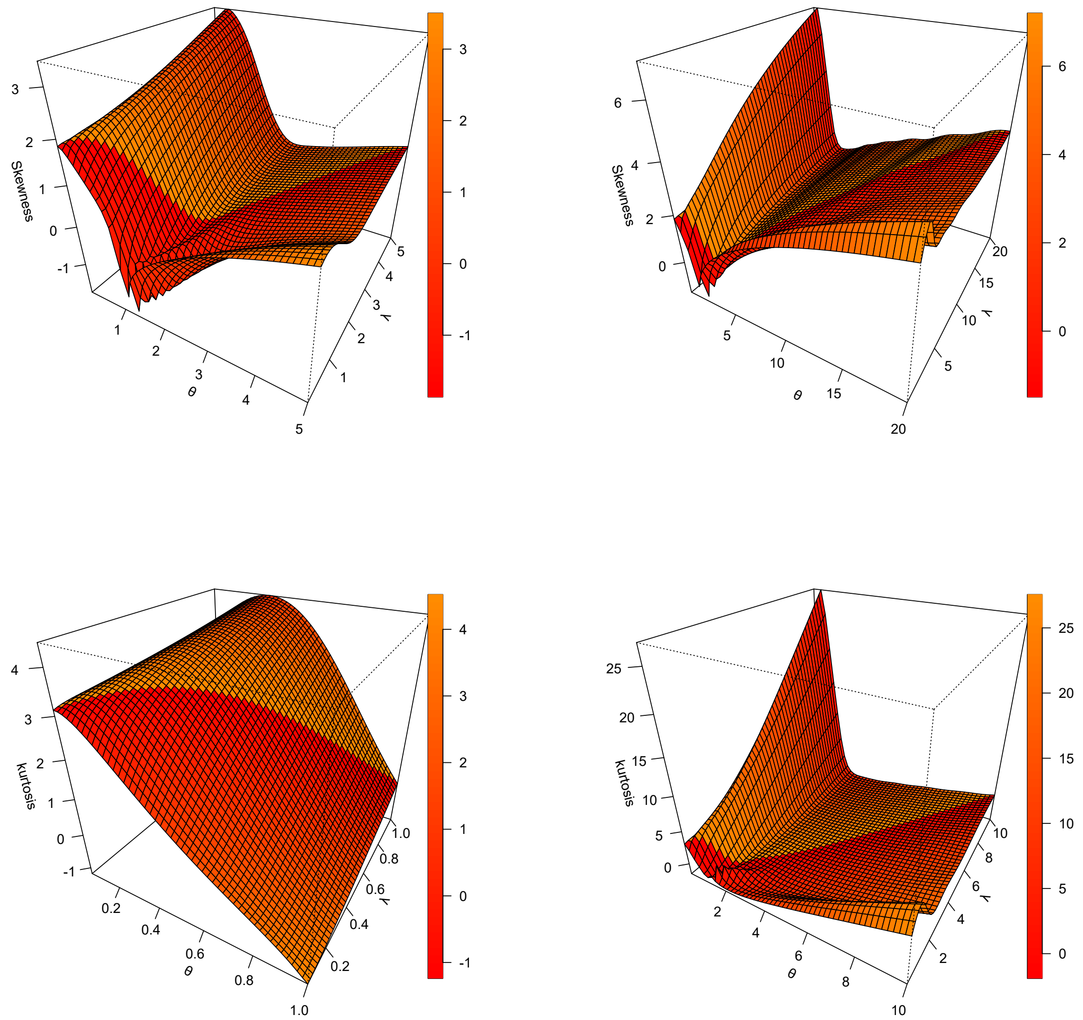

Fig. 4. Skewness and Kurtosis of ARS-W(θ, λ, ω)

Figure (4), the skewness and kurtosis of ARS-W(θ, λ, ω) distribution are presented for various values of parameters. The top plots demonstrate that the distribution is positively and negatively skewed; the top-left plot of parameters 1-5 evidently demonstrates a transition from negative/near-zero skewness to extremely positive skewness with increasing parameters. The upper-right plot (parameters 5-20) has consistently positive and sharply increasing skewness, exhibiting an obvious right-tailed nature at higher levels of parameters. The lower plots, which address the issue of kurtosis, highlight its responsiveness to changing parameters. The lower-left plot (parameters 0.2-1.0) reports an overall increase in kurtosis with higher parameters, exhibiting a more peaked distribution with thicker tails. Most notably, the lower-right plot (parameters 0-10) demonstrates a wild "spike" of kurtosis up to 25 when either or both parameters are set at their lower boundaries (nearly 0 or 1). This indicates that the ARS-W distribution can have wild leptokurtosis and, therefore, very peaked and heavy-tailed distributions, particularly when the parameter value is small. This illustrates just how adaptable the ARS-W distribution is towards asymmetry and tail behavior, as parameter selection is critical in that it determines these, especially in terms of regulating extreme values through kurtosis.

In general, Figure (4) displays the intricate parameter-shape property relationship of the ARS-W distribution. The skewness is either negative or positive, showing the flexibility of the distribution with respect to tail asymmetry. Essentially, kurtosis has a high sensitivity at the lower end of certain ranges of some parameters and can be very high. This implies that parameter selection plays a very vital role in deciding peakedness and heaviness of tail of ARS-W distribution, which is crucial for modeling events in which extreme events are a concern.

4.2 Some properties of the ARS-W distribution

The general results obtained for the ARS-G class of distributions are applied here to help us study some useful properties of the ARS-W distribution.

TABLE I. Some Properties of the ARS-W Distribution

Measures	Explicit Expressions
Noncentral Moments($\mu_r^{\prime}$)	$\sum_{i}^{\infty}\sum_{j}^{\infty}\sum_{n}^{\infty} w_i d_{k,m}\binom{2(j+k)+i-1}{n}\frac{\lambda\Gamma\left(\frac{r}{\omega}+1\right)}{(\lambda(n+1))^{\frac{r}{\omega}+1}}$
Mgf	$\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m w_i d_{2k+1,i-1}}{m!}\binom{2(j+k)+i-1}{n}\frac{\lambda\Gamma\left(\frac{r}{\omega}+1\right)}{(\lambda(n+1))^{\frac{r}{\omega}+1}}$
Order Statistic	$B_{i:n}\lambda\omega\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} Z_{j,k,l}b_r(-1)^p\binom{2(m+r)+k+l}{p}x^{\omega-1}e^{-\lambda(p+1)x^{\omega}}$
Entropy	$\frac{1}{1-s}\log\left(\sum_{v=1}^{\infty}\sum_{w=0}^{\infty}\sum_{q=0}^{\infty} t_v d_w(-1)^q\binom{2w+v-1}{q}\frac{\lambda^s\omega^{s-1}\Gamma(\frac{\omega s-s+1}{\omega})}{(\lambda(q+1))^{\frac{\omega s-s+1}{\omega}}}\right)$

Here $\Gamma(.) = \int_0^{\infty} t^{n-1}e^{-t}dt$ is the standard gamma function.

TABLE II. Numerical Results of Some Properties of ARS-W Model

θ	λ	ω	Mean	Variance	Std. Dev.	CV	Skewness	Kurtosis	Entropy	$MGF_{t=0.1}$	Order Statistic $_{minimum}$	Order Statistic $_{maximum}$
1.5	0.5	2	0.86966	0.21032	0.45860	0.52733	0.66028	3.30852	0.58216	1.09202	0.01428	3.37510
1.5	0.5	0.8	1.08841	1.94268	1.39380	1.28059	2.89070	16.45314	1.03442	1.12763	0.00000	21.08173
1.5	1	2	0.61494	0.10516	0.32428	0.52733	0.66028	3.30852	0.23558	1.06399	0.00530	2.09646
0.5	0.5	2	0.60988	0.15702	0.39626	0.64974	1.18865	4.71346	0.34315	1.06373	0.00770	2.90763
3	0.8	1.5	0.82116	0.23514	0.48492	0.59053	0.77866	3.66184	0.61450	1.08687	0.00181	3.42106

Table (II) numerically details the ARS-Weibull (ARS-W) distribution's properties across five parameter sets (θ, λ, ω), highlighting how parameter changes influence its characteristics. Regarding central tendency and dispersion, the Mean, Variance, and Standard Deviation (Std. Dev.) show typical values and spread. For example, increasing λ from 0.5 to 1 (comparing row 1 and 3) decreases both mean and variance. The Coefficient of Variation (CV) offers a normalized measure of variability; the identical CV for the first and third sets (0.52733) indicates proportional relative spread. The distribution's shape is revealed by Skewness and Kurtosis. All sets exhibit positive skewness, indicating right-tailed distributions. Notably, the second set ($\theta = 1.5, \lambda = 0.5, \omega = 0.8$) has a significantly higher skewness (2.89070), implying a much longer right tail. All distributions are leptokurtic (kurtosis > 3), signifying heavier tails and sharper peaks than a normal distribution. The second set's exceptionally high kurtosis (16.45314) further emphasizes its pronounced peakedness and very heavy tails. Entropy, measuring uncertainty, correlates with dispersion. The second parameter set, with its large variance and heavy tails, correspondingly shows the highest entropy (1.03442), reflecting its high uncertainty. The Order Statistics (minimum and maximum) provide an empirical range. The second set's maximum value (21.08173) is considerably larger than others, empirically confirming its heavy right tail and high variability. In essence, the table demonstrates the ARS-Weibull model's flexibility. Adjusting its parameters, particularly ω (as seen in the second set), can dramatically alter the distribution's mean, spread, and tail behavior, allowing it to model diverse phenomena, including those with extreme values.

5. PARAMETER ESTIMATION

This section explores fifteen distinct non-Bayesian estimation methods for determining the unknown parameters $(\theta, \lambda, \omega)$ of the ARS-W distribution. These methods are broadly classified into likelihood-based, distance-based, and spacing-based approaches. They include the Maximum Likelihood (MLE), Anderson-Darling (ADE), Cramér-von Mises (CVME), Maximum Product of Spacings (MPSE), Ordinary Least Squares (OLSE), Right-Tail Anderson-Darling (RTADE), Weighted Least Squares (WLSE), Left-Tail Anderson-Darling (LTADE), Minimum Spacing Absolute Distance (MSADE), Minimum Spacing Absolute-Log Distance (MSALDE), Anderson-Darling Second-Order Left-Tail (ADSOE), Kolmogorov (KE), Minimum Spacing Square Distance (MSSDE), Minimum Spacing Squared-Log Distance (MSSLDE), and Minimum Spacing Linex Distance (MSLNDE) estimators. The objective functions for each approach are detailed below.

5.1 Maximum Likelihood Estimation (MLE)

The Maximum Likelihood method, whose foundations were established by Fisher [58] and Fisher [59], is widely used due to its favorable asymptotic properties, such as consistency and asymptotic normality. Given a random sample $x_1, \ldots, x_n$ of size n from the ARS-W distribution with PDF $g(x; \theta, \lambda, \omega)$, the likelihood function $L(\theta, \lambda, \omega)$ is the product of the PDF evaluated at each observation.

$$\mathcal{L}(\theta, \lambda, \omega|\mathbf{x}) = \prod_{i=1}^{n} \left[\frac{\pi\theta\lambda\omega \cos\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right) x_i^{\omega-1} e^{-\lambda x_i^{\omega}}}{(\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right))^2} \right]$$

It is standard practice to work with the log-likelihood function, $\ell(\theta, \lambda, \omega|\mathbf{x}) = \ln(\mathcal{L}(\theta, \lambda, \omega|\mathbf{x}))$, because it converts the product into a sum, which is mathematically easier to handle. Since the logarithm is a monotonically increasing function, maximizing the log-likelihood is equivalent to maximizing the likelihood function.

$$\ell(\theta, \lambda, \omega|\mathbf{x}) = \sum_{i=1}^{n} \ln \left[\frac{\pi\theta\lambda\omega \cos\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right) x_i^{\omega-1} e^{-\lambda x_i^{\omega}}}{(\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right))^2} \right]$$

Using logarithm properties ($\ln(a/b) = \ln(a) - \ln(b)$ and $\ln(abc) = \ln(a) + \ln(b) + \ln(c)$), we get:

$$\begin{aligned}\ell(\theta, \lambda, \omega|\mathbf{x}) = \sum_{i=1}^{n} \Bigg[& \ln(\pi\theta\lambda\omega) + \ln\left(\cos\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right) + (\omega - 1)\ln(x_i) \\ & - \lambda x_i^{\omega} - 2\ln\left(\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right)\Bigg]\end{aligned}$$

Separating the sums and rearranging:

$$\begin{aligned}\ell(\theta, \lambda, \omega|\mathbf{x}) = {} & n\ln(\pi) + n\ln(\theta) + n\ln(\lambda) + n\ln(\omega) + (\omega - 1)\sum_{i=1}^{n}\ln(x_i) - \lambda\sum_{i=1}^{n} x_i^{\omega} \\ & + \sum_{i=1}^{n}\ln\left(\cos\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right) - 2\sum_{i=1}^{n}\ln\left(\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right)\end{aligned}$$

The Maximum Likelihood Estimators $(\hat{\theta}, \hat{\lambda}, \hat{\omega})$ are the values that maximize $\ell(\theta, \lambda, \omega|\mathbf{x})$. To find them, we must take the partial derivatives of the log-likelihood function with respect to each parameter and set them equal to zero (the likelihood equations).

Due to the highly nonlinear and coupled nature of the log-likelihood function, the likelihood equations will not yield closed-form expressions for the MLEs. They must be solved numerically.

The equations are:

$$\frac{\partial \ell}{\partial \theta} = 0.$$

$$\frac{\partial \ell}{\partial \lambda} = 0.$$

$$\frac{\partial \ell}{\partial \omega} = 0.$$

Partial derivative with respect to θ:

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n} \frac{1 - \sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)}{\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)} = 0$$

Partial derivative with respect to λ:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\omega} + \sum_{i=1}^{n}\left[\frac{\partial}{\partial \lambda}\ln\left(\cos\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right)\right] - 2\sum_{i=1}^{n}\left[\frac{\partial}{\partial \lambda}\ln\left(\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right)\right] = 0$$

Partial derivative with respect to ω:

$$\frac{\partial \ell}{\partial \omega} = \frac{n}{\omega} + \sum_{i=1}^{n}\ln(x_i) - \lambda\sum_{i=1}^{n} x_i^{\omega}\ln(x_i) + \sum_{i=1}^{n}\left[\frac{\partial}{\partial \omega}\ln\left(\cos\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right)\right] - 2\sum_{i=1}^{n}\left[\frac{\partial}{\partial \omega}\ln\left(\theta + (2-\theta)\sin\left(\frac{\pi}{2}(1 - e^{-\lambda x_i^{\omega}})\right)\right)\right] = 0$$

The Maximum Likelihood Estimators $(\hat{\theta}, \hat{\lambda}, \hat{\omega})$ are the simultaneous solutions to the system of equations formed by setting the partial derivatives of the log-likelihood function to zero. Due to the complexity of the equations, these estimators are found through numerical optimization techniques (e.g., Newton-Raphson or other gradient-based methods) using statistical software.

5.2 Estimation Methods Based on CDF

5.2.1. Anderson-Darling Estimation (ADE)

The ADE approach, pioneered by [60] and [61], minimizes a weighted statistic that places greater weight on the tails of the distribution. Parameter estimates for the ARS-W model are derived by minimizing the Anderson-Darling statistic, $A(\theta, \lambda, \omega)$:

$$A(\theta, \lambda, \omega) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n}) + \log S(x_{n-i+1:n})\right],$$

where $x_{i:n}$ are the order statistics, $F(x)$ is the CDF, and $S(x) = 1 - F(x)$ is the survival function.

5.2.2. Cramér-von Mises Estimation (CVME)

The CVME approach, applied to estimation by [62, 63, 64], minimizes a criterion sensitive to discrepancies across the entire distribution range. The ARS-W parameters $\hat{\theta}, \hat{\lambda}, \hat{\omega}$ are obtained by minimizing the Cramér-von Mises statistic, C:

$$C(\theta, \lambda, \omega) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2.$$

5.2.3. Ordinary Least Squares Estimation (OLSE)

The OLSE technique, utilized in distributional fitting by [65, 66], minimizes the sum of squared differences between the empirical and theoretical CDFs. The ARS-W parameters are estimated by minimizing the Ordinary Least Squares statistic, V:

$$V(\theta, \lambda, \omega) = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2.$$

5.2.4. Right-Tail Anderson-Darling Estimation (RTADE)

The RTADE method prioritizes goodness-of-fit in the upper (right) tail of the distribution. The parameters $\hat{\theta}, \hat{\lambda}, \hat{\omega}$ are found by minimizing the RTADE statistic, R:

$$R(\theta, \lambda, \omega) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{i:n}).$$

5.2.5. Weighted Least Squares Estimation (WLSE)

The WLSE method [65, 67] is a robust variant of OLSE that uses weights, typically reducing the influence of extreme observations. The ARS-W parameter estimates are derived by minimizing the Weighted Least Squares statistic, W:

$$W(\theta, \lambda, \omega) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2.$$

5.2.6. Left-Tail Anderson-Darling Estimation (LTADE)

The LTADE method focuses on optimizing the fit in the lower (left) tail of the distribution, based on procedures such as those detailed in [68]. The ARS-W parameters $\hat{\theta}, \hat{\lambda}, \hat{\omega}$ are estimated by minimizing the LTADE statistic, L:

$$L(\theta, \lambda, \omega) = -\frac{3}{2}n + 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(x_{i:n}).$$

5.2.7. Anderson-Darling Second-Order Left-Tail (ADSOE)

This refined technique, discussed in [69, 70], investigates the second-order behavior in the left tail. The ARS-W parameters are estimated by minimizing the ADSOE statistic, LTS:

$$LTS(\theta, \lambda, \omega) = 2\sum_{i=1}^{n}\log F(x_i) + \frac{1}{n}\sum_{i=1}^{n}\frac{(2i-1)}{F(x_i)}.$$

5.2.8. Kolmogorov Estimation (KE)

The Kolmogorov method, also employed in studies like [69], minimizes the maximum distance between the empirical and theoretical CDFs. The ARS-W parameters are estimated by minimizing the Kolmogorov statistic, KM:

$$KM(\theta, \lambda, \omega) = \underset{1\le i\le n}{\mathrm{MAX}}\left[\frac{i}{n} - F(x_i), F(x_i) - \frac{i-1}{n}\right].$$

5.3 Estimation Methods Based on Product of Spacings (PS)

These methods rely on the uniform spacings $I_i = F(x_{i:n}) - F(x_{i-1:n})$, where $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$.

5.3.1. Maximum Product of Spacings Estimation (MPSE)

The MPSE method, proposed by [71, 72], is a nonparametric alternative to MLE, suitable when the likelihood function is problematic. It maximizes the geometric mean of the spacings, meaning the ARS-W parameters are found by minimizing the negative log-product of spacings:

$$\delta(\theta, \lambda, \omega) = -\frac{1}{n+1}\sum_{i=1}^{n+1}\log I_i(\theta, \lambda, \omega).$$

5.3.2. Minimum Spacing Absolute Distance (MSADE)

The MSADE approach, introduced by [73], minimizes the sum of absolute differences between the observed spacings and their expected value, $1/(n+1)$. The MSADE statistic, ζ, is:

$$\zeta(\theta, \lambda, \omega) = \sum_{i=1}^{n+1}\left|I_i - \frac{1}{n+1}\right|.$$

5.3.3. Minimum Spacing Absolute-Log Distance (MSALDE)

The MSALDE method, also from [73], uses a logarithmic transformation of the spacings to estimate parameters. The ARS-W parameters are estimated by minimizing the MSALDE statistic, Υ:

$$\Upsilon(\theta, \lambda, \omega) = \sum_{i=1}^{n+1} \left| \log I_i - \log \frac{1}{n+1} \right|.$$

5.3.4. Minimum Spacing Square Distance (MSSDE)

The MSSDE method employs the sum of squared differences between the observed spacings and their expected uniform value. The ARS-W parameters are estimated by minimizing the MSSDE statistic, ϕ:

$$\phi(\theta, \lambda, \omega) = \sum_{i=1}^{n+1} \left(I_i - \frac{1}{n+1} \right)^2.$$

5.3.5. Minimum Spacing Squared-Log Distance (MSSLDE)

The MSSLDE approach utilizes the squared differences of the log-transformed spacings. The ARS-W parameters are derived by minimizing the MSSLDE statistic, Ψ:

$$\Psi(\theta, \lambda, \omega) = \sum_{i=1}^{n+1} \left(\log I_i - \log \frac{1}{n+1} \right)^2.$$

5.3.6. Minimum Spacing Linex Distance (MSLNDE)

The MSLNDE technique employs an asymmetric Linex-type loss function for parameter estimation. The ARS-W parameters are estimated by minimizing the MSLNDE statistic, $\mathit{\Delta}$:

$$\mathit{\Delta}(\theta, \lambda, \omega) = \sum_{i=1}^{n+1} \left[e^{I_i - \frac{1}{n+1}} - \left(I_i - \frac{1}{n+1} \right) - 1 \right].$$

6. SIMULATION

This section evaluates the efficacy of several estimation techniques in predicting the parameters of the ARS-W distribution using a substantial amount of simulated data. Random data sets were created for various sample sizes ($n = 15, 50, 120, 180, 250,$ and 350) using the quantile function of the suggested model in the simulation run. This section will evaluate the efficacy and performance of our model estimators. Furthermore, we will assess the effectiveness of several estimation methods considering many aspects. The evaluation will focus on the accuracy, precision, and computing efficiency of each method. By conducting a comprehensive examination of data from diverse sample sizes, our objective was to provide significant insight into the most reliable estimation approach for the NGMTE distribution. In addition, we will evaluate the efficacy of each estimating approach in various settings to determine its dependability and applicability in actual contexts. This research will help researchers select the most suitable estimation approach for their specific needs. Furthermore, we will assess the effectiveness of various estimation methodologies, taking into account several elements such as

$$\begin{cases} \text{Bias :} & |Bias(\widehat{\boldsymbol{\zeta}})| & = \frac{1}{D}\sum_{i=1}^{D} |\widehat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}|, \\ \text{Mean Squared Errors :} & \text{MSE} & = \frac{1}{D}\sum_{i=1}^{D} (\widehat{\boldsymbol{\zeta}} - \boldsymbol{\zeta})^2, \\ \text{Mean Relative Errors :} & \text{MRE} & = \frac{1}{D}\sum_{i=1}^{D} |\widehat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}|/\boldsymbol{\zeta}, \\ \text{Average Absolute Difference :} & D_{\text{abs}} & = \frac{1}{n\,D}\sum_{i=1}^{D}\sum_{j=1}^{n} |G(x_{ij}|\boldsymbol{\zeta}) - G(x_{ij}|\widehat{\boldsymbol{\zeta}})|, \\ \text{Maximum Absolute Difference :} & D_{\max} & = \frac{1}{D}\sum_{i=1}^{D} \max_j |G(x_{ij}|\boldsymbol{\zeta}) - G(x_{ij}|\widehat{\boldsymbol{\zeta}})|, \end{cases}$$

where the parameter vector is $\boldsymbol{\zeta} = (\theta,\ \lambda,\ \omega)$.

The simulation results in Table III provide a clear comparison of the 15 estimators, judged primarily by the sum of ranks ($\sum$ Ranks), where a lower value indicates a better overall performance across all parameters and metrics. For smaller sample sizes, the results suggest that the methods that minimize specific deviation measures are most effective. Specifically, MSADE and MSALDE secured the highest ranks at $n = 15$. As the sample size increased to $n = 50$ and $n = 120$, the Maximum Likelihood Estimator (MLE) became highly competitive, occasionally taking the top rank, but MSADE and ADE remained strong alternatives. The dominance of MLE is confirmed with larger sample sizes. It achieves the lowest overall rank at $n = 250$ and demonstrates overwhelming superiority at $n = 350$ with a $\sum$ Ranks of $12^{\{1\}}$, solidifying its expected asymptotic efficiency. In contrast, KE and methods like MSLNDE consistently occupy the lowest ranks across all sample sizes, indicating that they are the least suitable choices for estimating these parameters under the given simulation conditions. Overall, while MLE is the clear winner for large data, MSADE and MSALDE provide the most effective and stable results when the sample size is limited.

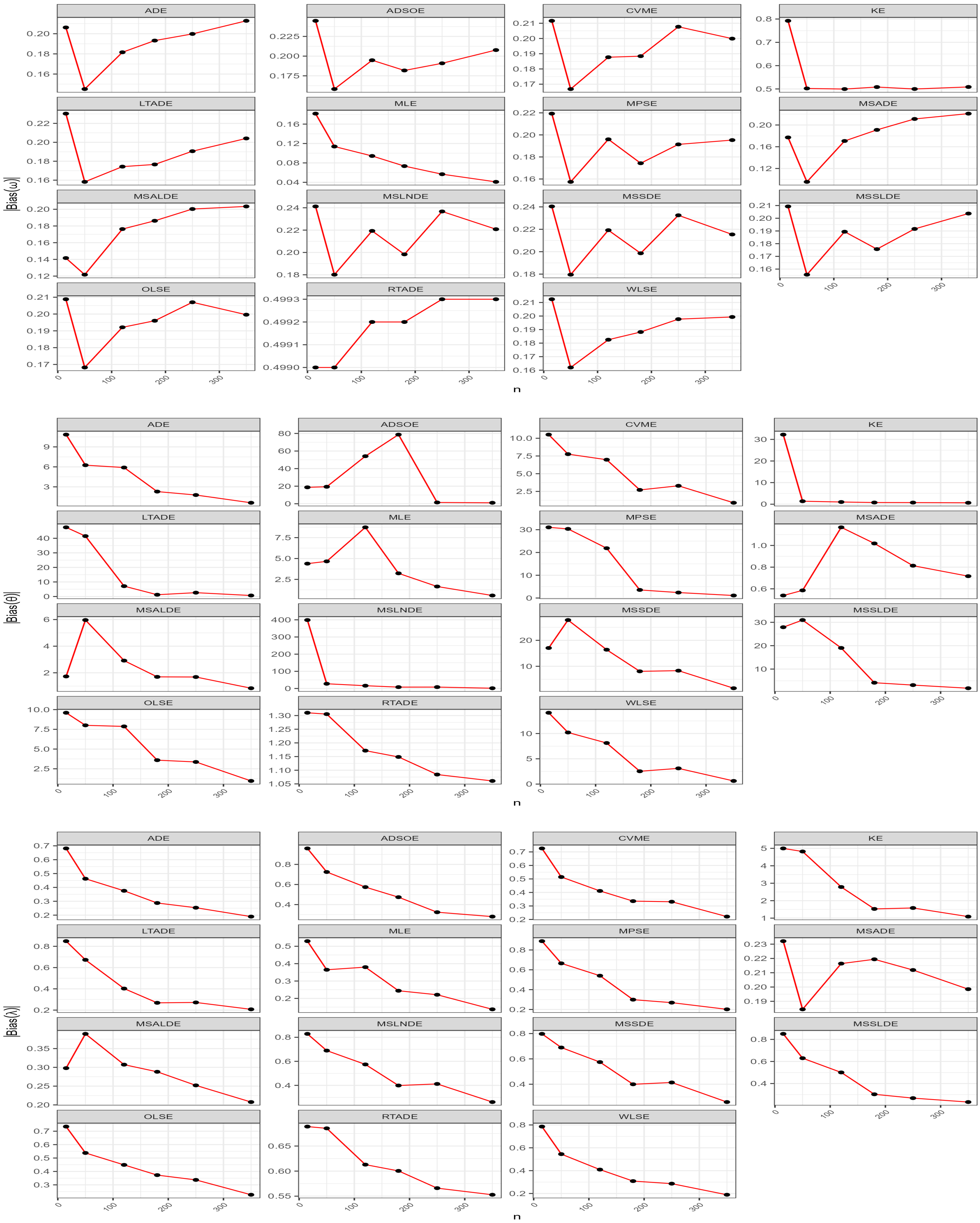

Fig. 5. Graphical representations for the Bias values presented in Table III.

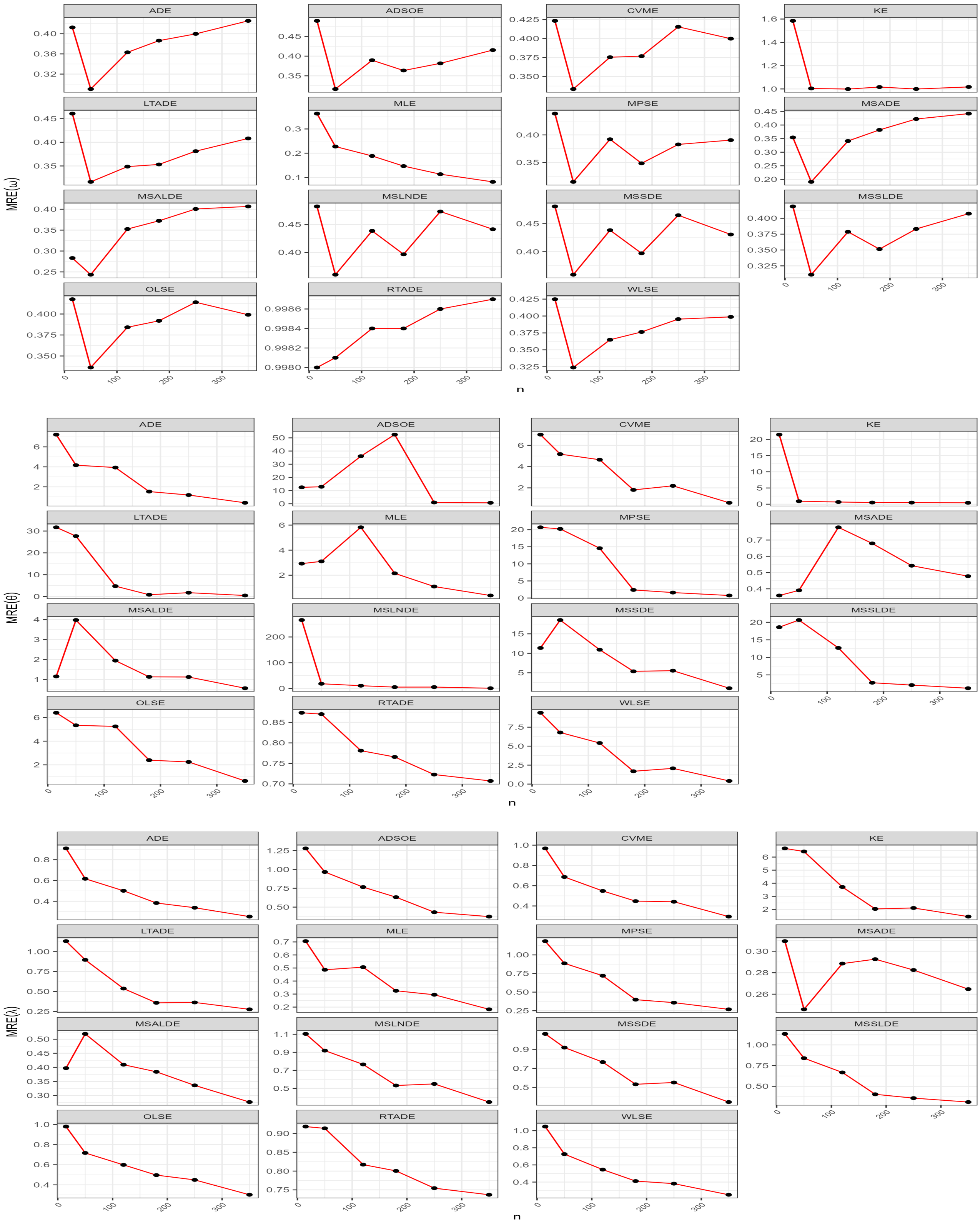

Fig. 6. Graphical representations for the MRE values presented in Table III.

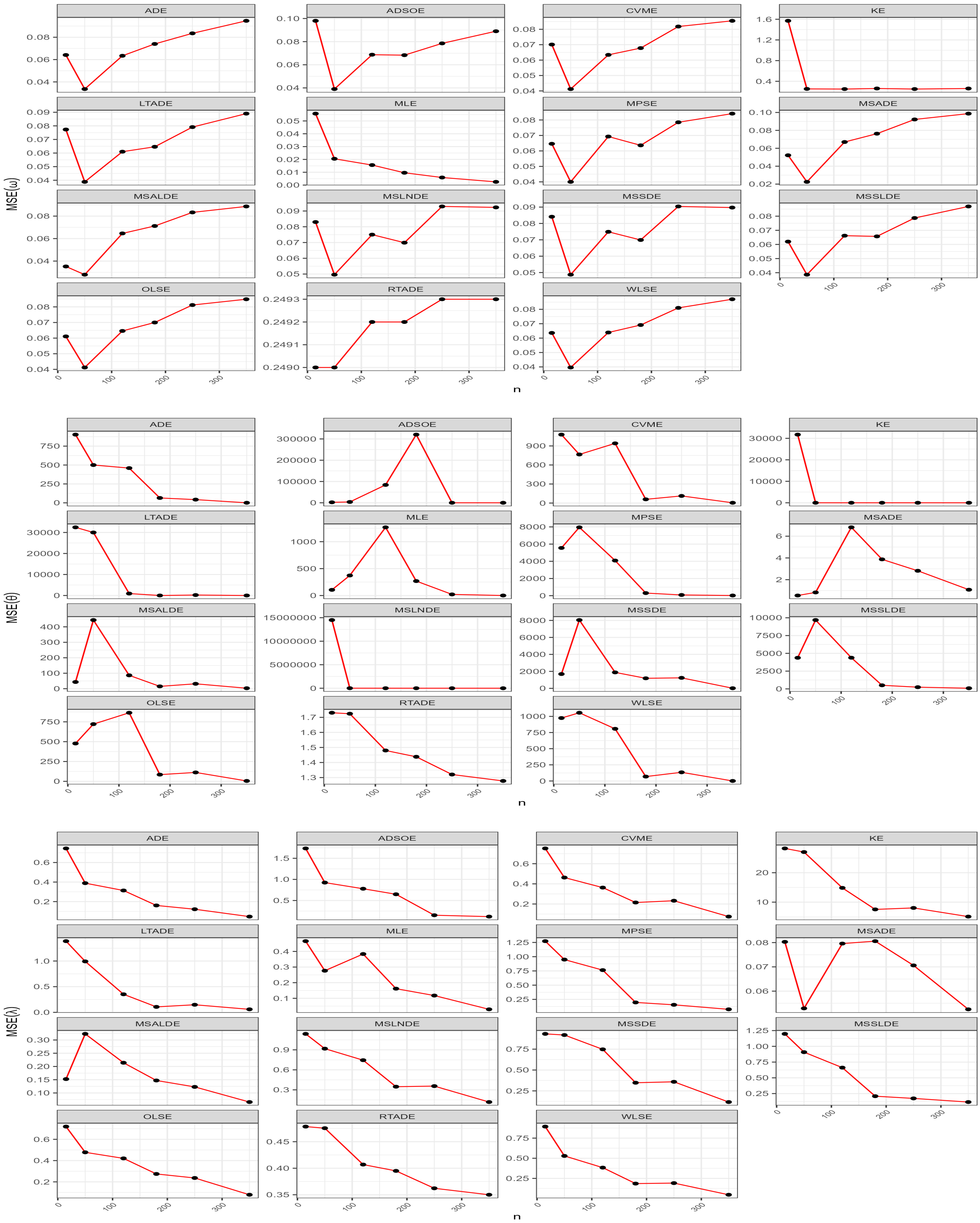

Fig. 7. Graphical representations for the MSE values presented in Table III.

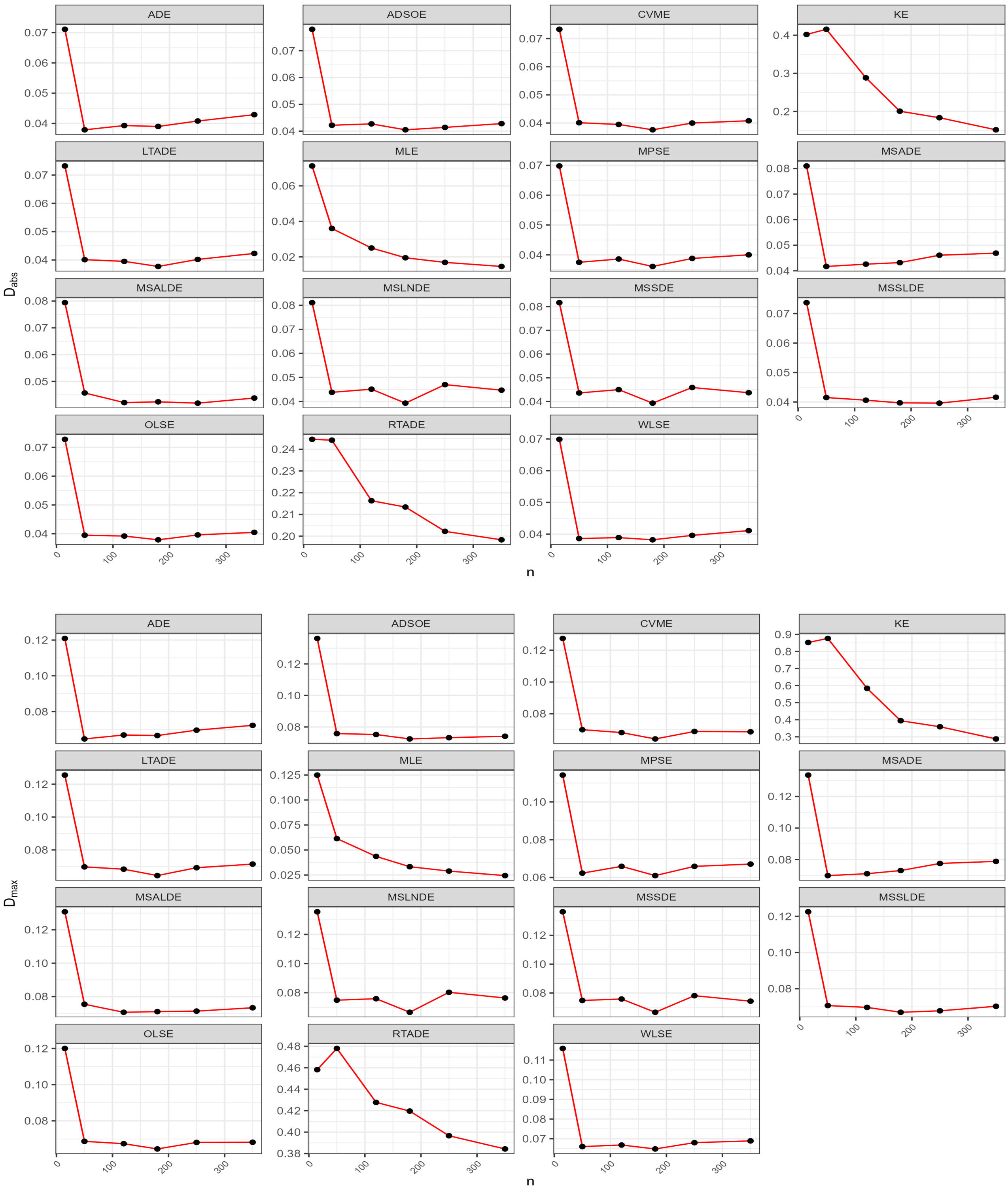

Fig. 8. Graphical representations for the D_{abs} and D_{max} values presented in Table III.

TABLE III. Numerical values of simulation measures for $\theta = 1.5,\ \lambda = 0.75\ \omega = 0.5$.

n	Metric	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
15	$\|Bias\|(\hat{\theta})$	4.3844[4]	10.8425[7]	10.5165[6]	31.0408[12]	9.5977[5]	1.3102[2]	14.1029[8]	47.5431[14]	0.5385[1]	1.7272[3]	18.6933[10]	32.2147[13]	17.0503[9]	27.9019[11]	398.6524[15]
	$\|Bias\|(\hat{\lambda})$	0.5295[3]	0.6812[4]	0.7258[6]	0.8887[13]	0.7344[7]	0.6888[5]	0.7857[8]	0.8486[11]	0.2321[1]	0.2979[2]	0.959[14]	4.9952[15]	0.7975[9]	0.8501[12]	0.8285[10]
	$\|Bias\|(\hat{\omega})$	0.1816[3]	0.2062[4]	0.2116[7]	0.2193[9]	0.2088[5]	0.499[14]	0.2124[8]	0.2303[10]	0.1771[2]	0.1416[1]	0.2448[13]	0.7923[15]	0.2404[11]	0.2093[6]	0.2412[12]
	MSE ($\hat{\theta}$)	105.3653[4]	902.2018[6]	1078.4613[8]	5553.2269[12]	477.2078[5]	1.7304[2]	974.2854[7]	32450.2248[14]	0.5437[1]	43.3617[3]	2240.0332[10]	31735.5803[13]	1684.9674[9]	4373.738[11]	14503313.7108[15]
	MSE ($\hat{\lambda}$)	0.4658[3]	0.7443[6]	0.7519[7]	1.2729[12]	0.7197[5]	0.4785[4]	0.8932[8]	1.3872[13]	0.0803[1]	0.1524[2]	1.7336[14]	28.2628[15]	0.9331[9]	1.1978[11]	1.1395[10]
	MSE ($\hat{\omega}$)	0.0557[3]	0.0641[7]	0.0701[9]	0.0646[8]	0.0611[4]	0.249[14]	0.0636[6]	0.0773[10]	0.0522[2]	0.0352[1]	0.098[13]	1.5698[15]	0.0841[12]	0.062[5]	0.083[11]
	MRE ($\hat{\theta}$)	2.9229[4]	7.2283[7]	7.011[6]	20.6938[12]	6.3984[5]	0.8735[2]	9.4019[8]	31.6954[14]	0.359[1]	1.1514[3]	12.4622[10]	21.4765[13]	11.3669[9]	18.6013[11]	265.7683[15]
	MRE ($\hat{\lambda}$)	0.7061[3]	0.9082[4]	0.9678[6]	1.1849[13]	0.9792[7]	0.9184[5]	1.0475[8]	1.1314[11]	0.3094[1]	0.3972[2]	1.2787[14]	6.6602[15]	1.0633[9]	1.1335[12]	1.1046[10]
	MRE ($\hat{\omega}$)	0.3633[3]	0.4125[4]	0.4233[7]	0.4386[9]	0.4176[5]	0.998[14]	0.4248[8]	0.4605[10]	0.3543[2]	0.2832[1]	0.4897[13]	1.5847[15]	0.4807[11]	0.4187[6]	0.4825[12]
	D_{abs}	0.0712[4]	0.0711[3]	0.0733[7]	0.0698[1]	0.0728[5]	0.2446[14]	0.0699[2]	0.0732[6]	0.081[11]	0.0794[10]	0.078[9]	0.402[15]	0.0817[13]	0.0737[8]	0.0811[12]
	D_{max}	0.1249[6]	0.1209[4]	0.1274[8]	0.1142[1]	0.12[3]	0.4582[14]	0.1159[2]	0.1255[7]	0.1335[10]	0.1308[9]	0.1362[12]	0.8525[15]	0.1363[13]	0.1225[5]	0.1355[11]
	$\sum$Ranks	40[3]	56[4.5]	77[7]	102[10]	56[4.5]	90[8]	73[6]	120[12]	33[1]	37[2]	132[13]	159[15]	114[11]	98[9]	133[14]
50	$\|Bias\|(\hat{\theta})$	4.6594[4]	6.254[6]	7.7616[7]	30.3511[13]	8.0061[8]	1.3053[2]	10.2043[9]	41.5123[15]	0.5857[1]	5.9622[5]	19.3387[10]	1.3771[3]	27.8142[12]	30.9674[14]	27.5709[11]
	$\|Bias\|(\hat{\lambda})$	0.3648[2]	0.4625[4]	0.5144[5]	0.6648[9]	0.5379[6]	0.6853[11]	0.5447[7]	0.6727[10]	0.1844[1]	0.3892[3]	0.7245[14]	4.8195[15]	0.6896[13]	0.6297[8]	0.6893[12]
	$\|Bias\|(\hat{\omega})$	0.1138[2]	0.1451[4]	0.1668[10]	0.1574[6]	0.1682[11]	0.499[14]	0.162[9]	0.1582[7]	0.0954[1]	0.1217[3]	0.1582[8]	0.5022[15]	0.1795[12]	0.1556[5]	0.1801[13]
	MSE ($\hat{\theta}$)	372.518[4]	500.0901[6]	765.4751[8]	7950.1193[12]	721.0508[7]	1.724[2]	1055.9998[9]	29996.0025[15]	0.8272[1]	443.9026[5]	3881.173[10]	2.0042[3]	8044.2726[13]	9681.9181[14]	7816.2059[11]
	MSE ($\hat{\lambda}$)	0.2765[2]	0.3889[4]	0.4626[5]	0.9489[13]	0.4778[7]	0.4755[6]	0.5292[8]	0.9927[14]	0.0529[1]	0.3231[3]	0.9244[12]	27.0342[15]	0.9193[11]	0.9076[9]	0.9176[10]
	MSE ($\hat{\omega}$)	0.0205[1]	0.0336[4]	0.0412[11]	0.04[9]	0.0412[10]	0.249[14]	0.0396[8]	0.0388[6]	0.0225[2]	0.0279[3]	0.039[7]	0.2525[15]	0.0487[12]	0.0386[5]	0.0496[13]
	MRE ($\hat{\theta}$)	3.1063[4]	4.1693[6]	5.1744[7]	20.2341[13]	5.3374[8]	0.8702[2]	6.8028[9]	27.6749[15]	0.3905[1]	3.9748[5]	12.8924[10]	0.9181[3]	18.5428[12]	20.6449[14]	18.3806[11]
	MRE ($\hat{\lambda}$)	0.4864[2]	0.6166[4]	0.6859[5]	0.8864[9]	0.7173[6]	0.9137[11]	0.7263[7]	0.8969[10]	0.2459[1]	0.519[3]	0.9661[14]	6.426[15]	0.9195[13]	0.8396[8]	0.9191[12]
	MRE ($\hat{\omega}$)	0.2275[2]	0.2903[4]	0.3335[10]	0.3149[6]	0.3363[11]	0.9981[14]	0.3239[9]	0.3164[7]	0.1909[1]	0.2434[3]	0.3164[8]	1.0043[15]	0.359[12]	0.3111[5]	0.3602[13]
	D_{abs}	0.036[1]	0.0379[3]	0.0401[7]	0.0375[2]	0.0395[5]	0.2442[14]	0.0386[4]	0.0401[6]	0.0417[9]	0.0457[13]	0.0422[10]	0.4156[15]	0.0436[11]	0.0415[8]	0.0438[12]
	D_{max}	0.0614[1]	0.0647[3]	0.07[8]	0.0623[2]	0.0687[5]	0.478[14]	0.066[4]	0.0697[6]	0.0699[7]	0.0754[12]	0.0758[13]	0.8762[15]	0.0748[10]	0.0707[9]	0.0749[11]
	$\sum$Ranks	25[1]	48[3]	83[5.5]	94[8]	84[7]	104[10]	83[5.5]	111[11]	26[2]	58[4]	116[12]	129[13.5]	131[15]	99[9]	129[13.5]
120	$\|Bias\|(\hat{\theta})$	8.7335[10]	5.9007[5]	6.974[6]	21.8672[14]	7.8696[8]	1.1716[3]	8.1223[9]	7.1164[7]	1.1671[2]	2.9141[4]	54.1211[15]	1.0123[1]	16.3825[12]	19.0272[13]	16.2145[11]
	$\|Bias\|(\hat{\lambda})$	0.3799[4]	0.3756[3]	0.4112[7]	0.5403[10]	0.4489[8]	0.6129[14]	0.4091[6]	0.4026[5]	0.2164[1]	0.3072[2]	0.5741[11]	2.7848[15]	0.5747[13]	0.5006[9]	0.5743[12]
	$\|Bias\|(\hat{\omega})$	0.0943[1]	0.1816[5]	0.1877[7]	0.196[11]	0.1921[9]	0.4992[14]	0.1825[6]	0.1743[3]	0.1706[2]	0.1763[4]	0.1948[10]	0.4995[15]	0.2192[12]	0.1894[8]	0.2193[13]
	MSE ($\hat{\theta}$)	1266.6369[10]	459.6555[5]	940.6[9]	4085.2518[13]	864.2642[7]	1.4798[2]	807.5168[6]	929.9687[8]	6.8156[3]	86.3903[4]	83984.9318[15]	1.2741[1]	1879.7515[12]	4382.14[14]	1830.9705[11]
	MSE ($\hat{\lambda}$)	0.3834[6]	0.3137[3]	0.3629[5]	0.7641[13]	0.4208[9]	0.4069[8]	0.3838[7]	0.3518[4]	0.0796[1]	0.214[2]	0.7787[14]	14.8458[15]	0.7477[12]	0.6618[10]	0.7449[11]
	MSE ($\hat{\omega}$)	0.0156[1]	0.0634[3]	0.0634[4]	0.0693[11]	0.0646[7]	0.2492[14]	0.0639[5]	0.061[2]	0.067[9]	0.0646[6]	0.0687[10]	0.2495[15]	0.0749[12]	0.0662[8]	0.075[13]
	MRE ($\hat{\theta}$)	5.8223[10]	3.9338[5]	4.6493[6]	14.5781[14]	5.2464[8]	0.7811[3]	5.4148[9]	4.7442[7]	0.7781[2]	1.9428[4]	36.0807[15]	0.6749[1]	10.9217[12]	12.6848[13]	10.8097[11]
	MRE ($\hat{\lambda}$)	0.5065[4]	0.5009[3]	0.5482[7]	0.7204[10]	0.5986[8]	0.8172[14]	0.5455[6]	0.5368[5]	0.2885[1]	0.4095[2]	0.7655[11]	3.713[15]	0.7663[13]	0.6675[9]	0.7657[12]
	MRE ($\hat{\omega}$)	0.1886[1]	0.3631[5]	0.3754[7]	0.3921[11]	0.3842[9]	0.9984[14]	0.3649[6]	0.3486[3]	0.3412[2]	0.3526[4]	0.3896[10]	0.999[15]	0.4383[12]	0.3787[8]	0.4387[13]
	D_{abs}	0.025[1]	0.0393[5]	0.0395[7]	0.0386[2]	0.0392[4]	0.2163[14]	0.0389[3]	0.0395[6]	0.0426[10]	0.0421[9]	0.0427[11]	0.2881[15]	0.045[12]	0.0406[8]	0.0451[13]
	D_{max}	0.0436[1]	0.0669[4]	0.0682[6]	0.0659[2]	0.0674[5]	0.4277[14]	0.0668[3]	0.0683[7]	0.0711[10]	0.0706[9]	0.0752[11]	0.5839[15]	0.0758[12]	0.0697[8]	0.0759[13]
	$\sum$Ranks	49[3]	46[2]	71[7]	111[10]	82[8]	114[11]	66[6]	57[5]	43[1]	50[4]	133[13.5]	123[12]	134[15]	108[9]	133[13.5]
180	$\|Bias\|(\hat{\theta})$	3.2253[9]	2.2901[6]	2.7099[8]	3.5527[10]	3.5927[11]	1.1486[3]	2.5345[7]	1.2142[4]	1.0186[2]	1.6928[5]	78.6947[15]	0.7705[1]	8.057[14]	4.1606[12]	8.0352[13]
	$\|Bias\|(\hat{\lambda})$	0.244[2]	0.2875[4]	0.3357[9]	0.299[6]	0.3727[10]	0.6004[14]	0.3086[8]	0.2685[3]	0.2194[1]	0.2882[5]	0.4737[13]	1.5279[15]	0.3996[12]	0.3026[7]	0.3989[11]
	$\|Bias\|(\hat{\omega})$	0.0734[1]	0.1932[10]	0.1884[8]	0.1743[2]	0.196[11]	0.4992[14]	0.1882[7]	0.1766[4]	0.191[9]	0.1862[6]	0.1818[5]	0.5082[15]	0.1986[13]	0.1757[3]	0.1983[12]
	MSE ($\hat{\theta}$)	268.5294[10]	64.4683[7]	58.6031[6]	305.2828[11]	84.5213[9]	1.4379[2]	68.2951[8]	4.3761[4]	3.8703[3]	15.0417[5]	320361.4233[15]	0.7874[1]	1184.263[14]	505.4146[12]	1172.6033[13]
	MSE ($\hat{\lambda}$)	0.1617[5]	0.1606[4]	0.2148[9]	0.1966[7]	0.2744[10]	0.395[13]	0.1856[6]	0.1063[2]	0.0806[1]	0.1471[3]	0.648[14]	7.5286[15]	0.3475[12]	0.2085[8]	0.3473[11]
	MSE ($\hat{\omega}$)	0.0096[1]	0.074[12]	0.0678[5]	0.0636[2]	0.07[10]	0.2492[14]	0.0691[7]	0.0646[3]	0.0763[13]	0.0712[11]	0.0683[6]	0.2606[15]	0.0699[8]	0.0657[4]	0.0699[9]
	MRE ($\hat{\theta}$)	2.1502[9]	1.5268[6]	1.8066[8]	2.3685[10]	2.3951[11]	0.7657[3]	1.6897[7]	0.8095[4]	0.6791[2]	1.1285[5]	52.4631[15]	0.5137[1]	5.3713[14]	2.7737[12]	5.3568[13]
	MRE ($\hat{\lambda}$)	0.3253[2]	0.3833[4]	0.4476[9]	0.3987[6]	0.4969[10]	0.8005[14]	0.4115[8]	0.358[3]	0.2926[1]	0.3842[5]	0.6317[13]	2.0372[15]	0.5327[12]	0.4035[7]	0.5318[11]
	MRE ($\hat{\omega}$)	0.1468[1]	0.3863[10]	0.3768[8]	0.3485[2]	0.3919[11]	0.9984[14]	0.3764[7]	0.3532[4]	0.382[9]	0.3723[6]	0.3635[5]	1.0165[15]	0.3971[13]	0.3514[3]	0.3967[12]
	D_{abs}	0.0195[1]	0.039[7]	0.0376[3]	0.0361[2]	0.0379[5]	0.2134[15]	0.0382[6]	0.0377[4]	0.0432[13]	0.0424[12]	0.0405[11]	0.2005[14]	0.0393[8]	0.0397[10]	0.0393[9]
	D_{max}	0.0333[1]	0.0666[7]	0.0642[3]	0.061[2]	0.0645[5]	0.4196[15]	0.0648[6]	0.0644[4]	0.0731[13]	0.071[11]	0.0724[12]	0.3942[14]	0.0666[9]	0.067[10]	0.0666[8]
	$\sum$Ranks	42[2]	77[7.5]	76[6]	60[3]	103[10]	121[11.5]	77[7.5]	39[1]	67[4]	74[5]	124[14]	121[11.5]	129[15]	88[9]	122[13]
250	$\|Bias\|(\hat{\theta})$	1.6459[5]	1.7759[7]	3.291[12]	2.3978[8]	3.3664[13]	1.0842[3]	3.1074[10]	2.6346[9]	0.8134[2]	1.6831[6]	1.4574[4]	0.743[1]	8.3143[15]	3.1484[11]	8.185[14]
	$\|Bias\|(\hat{\lambda})$	0.221[2]	0.2535[4]	0.3312[10]	0.2689[6]	0.337[11]	0.566[14]	0.2861[8]	0.272[7]	0.2119[1]	0.252[3]	0.3241[9]	1.5824[15]	0.4138[13]	0.2679[5]	0.412[12]
	$\|Bias\|(\hat{\omega})$	0.0567[1]	0.1998[7]	0.2077[10]	0.1914[4]	0.207[9]	0.4993[14]	0.1977[6]	0.1906[2]	0.2111[11]	0.2003[8]	0.1909[3]	0.4998[15]	0.2325[12]	0.1916[5]	0.2367[13]
	MSE ($\hat{\theta}$)	21.03[5]	42.6095[7]	112.2194[10]	84.8071[8]	112.2192[9]	1.3202[2]	134.3015[11]	224.2081[12]	2.8165[3]	31.6729[6]	7.2247[4]	0.7358[1]	1237.2806[15]	240.9676[13]	1199.6852[14]
	MSE ($\hat{\lambda}$)	0.1177[2]	0.122[3]	0.2323[10]	0.1547[7]	0.2366[11]	0.3622[14]	0.1918[9]	0.1476[5]	0.0706[1]	0.1229[4]	0.1513[6]	8.0299[15]	0.3588[13]	0.1734[8]	0.3562[12]
	MSE ($\hat{\omega}$)	0.0059[1]	0.0835[10]	0.0818[8]	0.0785[2]	0.0812[7]	0.2493[14]	0.081[6]	0.0791[5]	0.0922[12]	0.0833[9]	0.0785[3]	0.2498[15]	0.0904[11]	0.0787[4]	0.0929[13]
	MRE ($\hat{\theta}$)	1.0973[5]	1.184[7]	2.194[12]	1.5986[8]	2.2443[13]	0.7228[3]	2.0716[10]	1.7564[9]	0.5423[2]	1.1221[6]	0.9716[4]	0.4953[1]	5.5429[15]	2.099[11]	5.4567[14]
	MRE ($\hat{\lambda}$)	0.2946[2]	0.338[4]	0.4416[10]	0.3585[6]	0.4493[11]	0.7546[14]	0.3815[8]	0.3627[7]	0.2825[1]	0.336[3]	0.4322[9]	2.1099[15]	0.5517[13]	0.3572[5]	0.5494[12]
	MRE ($\hat{\omega}$)	0.1134[1]	0.3997[7]	0.4154[10]	0.3828[4]	0.414[9]	0.9986[14]	0.3954[6]	0.3812[2]	0.4221[11]	0.4006[8]	0.3817[3]	0.9995[15]	0.465[12]	0.3832[5]	0.4734[13]
	D_{abs}	0.0169[1]	0.0408[8]	0.04[6]	0.0388[2]	0.0396[3]	0.2022[15]	0.0396[5]	0.0402[7]	0.0461[12]	0.0419[10]	0.0414[9]	0.1834[14]	0.0459[11]	0.0396[4]	0.047[13]
	D_{max}	0.0289[1]	0.0696[8]	0.0689[6]	0.0659[2]	0.0681[5]	0.3966[15]	0.068[4]	0.0692[7]	0.0776[11]	0.0713[9]	0.0732[10]	0.3589[14]	0.0781[12]	0.0678[3]	0.0803[13]
	$\sum$Ranks	26[1]	72[6]	104[11]	57[2]	101[10]	122[13]	83[9]	72[6]	67[4]	72[6]	64[3]	121[12]	142[14]	74[8]	143[15]
350	$\|Bias\|(\hat{\theta})$	0.5702[1]	0.6088[2]	0.8951[8]	1.1055[12]	0.9604[9]	1.0608[10]	0.6211[3]	0.6831[5]	0.7168[6]	0.8403[7]	1.0978[11]	0.6508[4]	1.5496[13]	1.8066[15]	1.5541[14]
	$\|Bias\|(\hat{\lambda})$	0.1371[1]	0.1894[3]	0.2212[8]	0.2016[5]	0.2262[9]	0.5528[14]	0.1889[2]	0.2074[6]	0.1985[4]	0.2076[7]	0.2804[13]	1.0876[15]	0.2594[11]	0.2323[10]	0.2613[12]
	$\|Bias\|(\hat{\omega})$	0.0409[1]	0.2128[10]	0.1999[5]	0.1953[2]	0.1996[4]	0.4993[14]	0.1993[3]	0.2041[8]	0.2209[13]	0.2034[6]	0.2077[9]	0.5086[15]	0.2154[11]	0.2037[7]	0.2208[12]
	MSE ($\hat{\theta}$)	0.5919[2]	0.6438[3]	4.1155[10]	21.9153[14]	4.9428[11]	1.2775[7]	0.6713[4]	0.948[5]	1.0756[6]	3.0491[8]	4.015[9]	0.5545[1]	14.535[13]	93.6728[15]	14.3586[12]
	MSE ($\hat{\lambda}$)	0.0295[1]	0.0464[2]	0.0747[7]	0.0755[8]	0.0787[9]	0.35[14]	0.0469[3]	0.0587[5]	0.0525[4]	0.0657[6]	0.1187[13]	5.0795[15]	0.1163[11]	0.1157[10]	0.1171[12]
	MSE ($\hat{\omega}$)	0.0025[1]	0.0947[12]	0.0855[4]	0.0841[2]	0.0849[3]	0.2493[14]	0.087[6]	0.089[8]	0.0987[13]	0.0886[7]	0.089[9]	0.2601[15]	0.0897[10]	0.0869[5]	0.0923[11]
	MRE ($\hat{\theta}$)	0.3802[1]	0.4058[2]	0.5967[8]	0.737[12]	0.6403[9]	0.7072[10]	0.4141[3]	0.4554[5]	0.4778[6]	0.5602[7]	0.7319[11]	0.4338[4]	1.033[13]	1.2044[15]	1.0361[14]
	MRE ($\hat{\lambda}$)	0.1828[1]	0.2525[3]	0.2949[8]	0.2688[5]	0.3016[9]	0.737[14]	0.2518[2]	0.2765[6]	0.2647[4]	0.2768[7]	0.3739[13]	1.4501[15]	0.3459[11]	0.3098[10]	0.3484[12]
	MRE ($\hat{\omega}$)	0.0817[1]	0.4256[10]	0.3998[5]	0.3906[2]	0.3991[4]	0.9987[14]	0.3987[3]	0.4081[8]	0.4419[13]	0.4067[6]	0.4154[9]	1.0173[15]	0.4307[11]	0.4073[7]	0.4416[12]
	D_{abs}	0.0146[1]	0.0429[9]	0.0408[4]	0.04[2]	0.0405[3]	0.1983[15]	0.0411[5]	0.0423[7]	0.0469[13]	0.0438[11]	0.0428[8]	0.1517[14]	0.0437[10]	0.0416[6]	0.0447[12]
	D_{max}	0.0244[1]	0.0723[8]	0.0687[4]	0.0671[2]	0.0682[3]	0.3843[15]	0.0689[5]	0.0714[7]	0.0789[13]	0.0733[9]	0.0741[10]	0.2876[14]	0.0743[11]	0.0703[6]	0.0764[12]
	$\sum$Ranks	12[1]	64[3]	71[6]	66[4]	73[7]	141[15]	39[2]	70[5]	95[8]	81[9]	115[11]	127[13]	125[12]	106[10]	135[14]

TABLE IV. Partial and overall ranks for all estimation methods of the ARS-W model from III.

n	Metric	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
15	$\|Bias\|(\hat{\theta})$	4.0	7.0	6.0	12.0	5.0	2.0	8.0	14.0	1.0	3.0	10.0	13.0	9.0	11.0	15.0
	$\|Bias\|(\hat{\lambda})$	3.0	4.0	6.0	13.0	7.0	5.0	8.0	11.0	1.0	2.0	14.0	15.0	9.0	12.0	10.0
	$\|Bias\|(\hat{\omega})$	3.0	4.0	7.0	9.0	5.0	14.0	8.0	10.0	2.0	1.0	13.0	15.0	11.0	6.0	12.0
	MSE $(\hat{\theta})$	4.0	6.0	8.0	12.0	5.0	2.0	7.0	14.0	1.0	3.0	10.0	13.0	9.0	11.0	15.0
	MSE $(\hat{\lambda})$	3.0	6.0	7.0	12.0	5.0	4.0	8.0	13.0	1.0	2.0	14.0	15.0	9.0	11.0	10.0
	MSE $(\hat{\omega})$	3.0	7.0	9.0	8.0	4.0	14.0	6.0	10.0	2.0	1.0	13.0	15.0	12.0	5.0	11.0
	MRE $(\hat{\theta})$	4.0	7.0	6.0	12.0	5.0	2.0	8.0	14.0	1.0	3.0	10.0	13.0	9.0	11.0	15.0
	MRE $(\hat{\lambda})$	3.0	4.0	6.0	13.0	7.0	5.0	8.0	11.0	1.0	2.0	14.0	15.0	9.0	12.0	10.0
	MRE $(\hat{\omega})$	3.0	4.0	7.0	9.0	5.0	14.0	8.0	10.0	2.0	1.0	13.0	15.0	11.0	6.0	12.0
	D_{abs}	4.0	3.0	7.0	1.0	5.0	14.0	2.0	6.0	11.0	10.0	9.0	15.0	13.0	8.0	12.0
	D_{max}	6.0	4.0	8.0	1.0	3.0	14.0	2.0	7.0	10.0	9.0	12.0	15.0	13.0	5.0	11.0
	$\sum$Ranks	40.0$^{(3)}$	56.0$^{(4)}$	77.0$^{(7)}$	102.0$^{(10)}$	56.0$^{(4)}$	90.0$^{(8)}$	73.0$^{(6)}$	120.0$^{(12)}$	33.0$^{(1)}$	37.0$^{(2)}$	132.0$^{(13)}$	159.0$^{(15)}$	114.0$^{(11)}$	98.0$^{(9)}$	133.0$^{(14)}$
50	$\|Bias\|(\hat{\theta})$	4.0	6.0	7.0	13.0	8.0	2.0	9.0	15.0	1.0	5.0	10.0	3.0	12.0	14.0	11.0
	$\|Bias\|(\hat{\lambda})$	2.0	4.0	5.0	9.0	6.0	11.0	7.0	10.0	1.0	3.0	14.0	15.0	13.0	8.0	12.0
	$\|Bias\|(\hat{\omega})$	2.0	4.0	10.0	6.0	11.0	14.0	9.0	7.0	1.0	3.0	8.0	15.0	12.0	5.0	13.0
	MSE $(\hat{\theta})$	4.0	6.0	8.0	12.0	7.0	2.0	9.0	15.0	1.0	5.0	10.0	3.0	13.0	14.0	11.0
	MSE $(\hat{\lambda})$	2.0	4.0	5.0	13.0	7.0	6.0	8.0	14.0	1.0	3.0	12.0	15.0	11.0	9.0	10.0
	MSE $(\hat{\omega})$	1.0	4.0	11.0	9.0	10.0	14.0	8.0	6.0	2.0	3.0	7.0	15.0	12.0	5.0	13.0
	MRE $(\hat{\theta})$	4.0	6.0	7.0	13.0	8.0	2.0	9.0	15.0	1.0	5.0	10.0	3.0	12.0	14.0	11.0
	MRE $(\hat{\lambda})$	2.0	4.0	5.0	9.0	6.0	11.0	7.0	10.0	1.0	3.0	14.0	15.0	13.0	8.0	12.0
	MRE $(\hat{\omega})$	2.0	4.0	10.0	6.0	11.0	14.0	9.0	7.0	1.0	3.0	8.0	15.0	12.0	5.0	13.0
	D_{abs}	1.0	3.0	7.0	2.0	5.0	14.0	4.0	6.0	9.0	13.0	10.0	15.0	11.0	8.0	12.0
	D_{max}	1.0	3.0	8.0	2.0	5.0	14.0	4.0	6.0	7.0	12.0	13.0	15.0	10.0	9.0	11.0
	$\sum$Ranks	25.0$^{(1)}$	48.0$^{(3)}$	83.0$^{(5)}$	94.0$^{(8)}$	84.0$^{(7)}$	104.0$^{(10)}$	83.0$^{(5)}$	111.0$^{(11)}$	26.0$^{(2)}$	58.0$^{(4)}$	116.0$^{(12)}$	129.0$^{(13)}$	131.0$^{(15)}$	99.0$^{(9)}$	129.0$^{(13)}$
120	$\|Bias\|(\hat{\theta})$	10.0	5.0	6.0	14.0	8.0	3.0	9.0	7.0	2.0	4.0	15.0	1.0	12.0	13.0	11.0
	$\|Bias\|(\hat{\lambda})$	4.0	3.0	7.0	10.0	8.0	14.0	6.0	5.0	1.0	2.0	11.0	15.0	13.0	9.0	12.0
	$\|Bias\|(\hat{\omega})$	1.0	5.0	7.0	11.0	9.0	14.0	6.0	3.0	2.0	4.0	10.0	15.0	12.0	8.0	13.0
	MSE $(\hat{\theta})$	10.0	5.0	9.0	13.0	7.0	2.0	6.0	8.0	3.0	4.0	15.0	1.0	12.0	14.0	11.0
	MSE $(\hat{\lambda})$	6.0	3.0	5.0	13.0	9.0	8.0	7.0	4.0	1.0	2.0	14.0	15.0	12.0	10.0	11.0
	MSE $(\hat{\omega})$	1.0	3.0	4.0	11.0	7.0	14.0	5.0	2.0	9.0	6.0	10.0	15.0	12.0	8.0	13.0
	MRE $(\hat{\theta})$	10.0	5.0	6.0	14.0	8.0	3.0	9.0	7.0	2.0	4.0	15.0	1.0	12.0	13.0	11.0
	MRE $(\hat{\lambda})$	4.0	3.0	7.0	10.0	8.0	14.0	6.0	5.0	1.0	2.0	11.0	15.0	13.0	9.0	12.0
	MRE $(\hat{\omega})$	1.0	5.0	7.0	11.0	9.0	14.0	6.0	3.0	2.0	4.0	10.0	15.0	12.0	8.0	13.0
	D_{abs}	1.0	5.0	7.0	2.0	4.0	14.0	3.0	6.0	10.0	9.0	11.0	15.0	12.0	8.0	13.0
	D_{max}	1.0	4.0	6.0	2.0	5.0	14.0	3.0	7.0	10.0	9.0	11.0	15.0	12.0	8.0	13.0
	$\sum$Ranks	49.0$^{(3)}$	46.0$^{(2)}$	71.0$^{(7)}$	111.0$^{(10)}$	82.0$^{(8)}$	114.0$^{(11)}$	66.0$^{(6)}$	57.0$^{(5)}$	43.0$^{(1)}$	50.0$^{(4)}$	133.0$^{(13)}$	123.0$^{(12)}$	134.0$^{(15)}$	108.0$^{(9)}$	133.0$^{(13)}$
180	$\|Bias\|(\hat{\theta})$	9.0	6.0	8.0	10.0	11.0	3.0	7.0	4.0	2.0	5.0	15.0	1.0	14.0	12.0	13.0
	$\|Bias\|(\hat{\lambda})$	2.0	4.0	9.0	6.0	10.0	14.0	8.0	3.0	1.0	5.0	13.0	15.0	12.0	7.0	11.0
	$\|Bias\|(\hat{\omega})$	1.0	10.0	8.0	2.0	11.0	14.0	7.0	4.0	9.0	6.0	5.0	15.0	13.0	3.0	12.0
	MSE $(\hat{\theta})$	10.0	7.0	6.0	11.0	9.0	2.0	8.0	4.0	3.0	5.0	15.0	1.0	14.0	12.0	13.0
	MSE $(\hat{\lambda})$	5.0	4.0	9.0	7.0	10.0	13.0	6.0	2.0	1.0	3.0	14.0	15.0	12.0	8.0	11.0
	MSE $(\hat{\omega})$	1.0	12.0	5.0	2.0	10.0	14.0	7.0	3.0	13.0	11.0	6.0	15.0	8.0	4.0	9.0
	MRE $(\hat{\theta})$	9.0	6.0	8.0	10.0	11.0	3.0	7.0	4.0	2.0	5.0	15.0	1.0	14.0	12.0	13.0
	MRE $(\hat{\lambda})$	2.0	4.0	9.0	6.0	10.0	14.0	8.0	3.0	1.0	5.0	13.0	15.0	12.0	7.0	11.0
	MRE $(\hat{\omega})$	1.0	10.0	8.0	2.0	11.0	14.0	7.0	4.0	9.0	6.0	5.0	15.0	13.0	3.0	12.0
	D_{abs}	1.0	7.0	3.0	2.0	5.0	15.0	6.0	4.0	13.0	12.0	11.0	14.0	8.0	10.0	9.0
	D_{max}	1.0	7.0	3.0	2.0	5.0	15.0	6.0	4.0	13.0	11.0	12.0	14.0	9.0	10.0	8.0
	$\sum$Ranks	42.0$^{(2)}$	77.0$^{(7)}$	76.0$^{(6)}$	60.0$^{(3)}$	103.0$^{(10)}$	121.0$^{(11)}$	77.0$^{(7)}$	39.0$^{(1)}$	67.0$^{(4)}$	74.0$^{(5)}$	124.0$^{(14)}$	121.0$^{(11)}$	129.0$^{(15)}$	88.0$^{(9)}$	122.0$^{(13)}$
250	$\|Bias\|(\hat{\theta})$	5.0	7.0	12.0	8.0	13.0	3.0	10.0	9.0	2.0	6.0	4.0	1.0	15.0	11.0	14.0
	$\|Bias\|(\hat{\lambda})$	2.0	4.0	10.0	6.0	11.0	14.0	8.0	7.0	1.0	3.0	9.0	15.0	13.0	5.0	12.0
	$\|Bias\|(\hat{\omega})$	1.0	7.0	10.0	4.0	9.0	14.0	6.0	2.0	11.0	8.0	3.0	15.0	12.0	5.0	13.0
	MSE $(\hat{\theta})$	5.0	7.0	10.0	8.0	9.0	2.0	11.0	12.0	3.0	6.0	4.0	1.0	15.0	13.0	14.0
	MSE $(\hat{\lambda})$	2.0	3.0	10.0	7.0	11.0	14.0	9.0	5.0	1.0	4.0	6.0	15.0	13.0	8.0	12.0
	MSE $(\hat{\omega})$	1.0	10.0	8.0	2.0	7.0	14.0	6.0	5.0	12.0	9.0	3.0	15.0	11.0	4.0	13.0
	MRE $(\hat{\theta})$	5.0	7.0	12.0	8.0	13.0	3.0	10.0	9.0	2.0	6.0	4.0	1.0	15.0	11.0	14.0
	MRE $(\hat{\lambda})$	2.0	4.0	10.0	6.0	11.0	14.0	8.0	7.0	1.0	3.0	9.0	15.0	13.0	5.0	12.0
	MRE $(\hat{\omega})$	1.0	7.0	10.0	4.0	9.0	14.0	6.0	2.0	11.0	8.0	3.0	15.0	12.0	5.0	13.0
	D_{abs}	1.0	8.0	6.0	2.0	3.0	15.0	5.0	7.0	12.0	10.0	9.0	14.0	11.0	4.0	13.0
	D_{max}	1.0	8.0	6.0	2.0	5.0	15.0	4.0	7.0	11.0	9.0	10.0	14.0	12.0	3.0	13.0
	$\sum$Ranks	26.0$^{(1)}$	72.0$^{(5)}$	104.0$^{(11)}$	57.0$^{(2)}$	101.0$^{(10)}$	122.0$^{(13)}$	83.0$^{(9)}$	72.0$^{(5)}$	67.0$^{(4)}$	72.0$^{(5)}$	64.0$^{(3)}$	121.0$^{(12)}$	142.0$^{(14)}$	74.0$^{(8)}$	143.0$^{(15)}$
350	$\|Bias\|(\hat{\theta})$	1.0	2.0	8.0	12.0	9.0	10.0	3.0	5.0	6.0	7.0	11.0	4.0	13.0	15.0	14.0
	$\|Bias\|(\hat{\lambda})$	1.0	3.0	8.0	5.0	9.0	14.0	2.0	6.0	4.0	7.0	13.0	15.0	11.0	10.0	12.0
	$\|Bias\|(\hat{\omega})$	1.0	10.0	5.0	2.0	4.0	14.0	3.0	8.0	13.0	6.0	9.0	15.0	11.0	7.0	12.0
	MSE $(\hat{\theta})$	2.0	3.0	10.0	14.0	11.0	7.0	4.0	5.0	6.0	8.0	9.0	1.0	13.0	15.0	12.0
	MSE $(\hat{\lambda})$	1.0	2.0	7.0	8.0	9.0	14.0	3.0	5.0	4.0	6.0	13.0	15.0	11.0	10.0	12.0
	MSE $(\hat{\omega})$	1.0	12.0	4.0	2.0	3.0	14.0	6.0	8.0	13.0	7.0	9.0	15.0	10.0	5.0	11.0
	MRE $(\hat{\theta})$	1.0	2.0	8.0	12.0	9.0	10.0	3.0	5.0	6.0	7.0	11.0	4.0	13.0	15.0	14.0
	MRE $(\hat{\lambda})$	1.0	3.0	8.0	5.0	9.0	14.0	2.0	6.0	4.0	7.0	13.0	15.0	11.0	10.0	12.0
	MRE $(\hat{\omega})$	1.0	10.0	5.0	2.0	4.0	14.0	3.0	8.0	13.0	6.0	9.0	15.0	11.0	7.0	12.0
	D_{abs}	1.0	9.0	4.0	2.0	3.0	15.0	5.0	7.0	13.0	11.0	8.0	14.0	10.0	6.0	12.0
	D_{max}	1.0	8.0	4.0	2.0	3.0	15.0	5.0	7.0	13.0	9.0	10.0	14.0	11.0	6.0	12.0
	$\sum$Ranks	12.0$^{(1)}$	64.0$^{(3)}$	71.0$^{(6)}$	66.0$^{(4)}$	73.0$^{(7)}$	141.0$^{(15)}$	39.0$^{(2)}$	70.0$^{(5)}$	95.0$^{(9)}$	81.0$^{(8)}$	115.0$^{(11)}$	127.0$^{(13)}$	125.0$^{(12)}$	106.0$^{(10)}$	135.0$^{(14)}$

Table IV indicates the partial ranks of some of the estimation procedures of the ARS-W model, condensing the data from the table above into a simpler rank-based comparison, with the sum of ranks (∑Ranks) being the main measure of overall estimator efficacy (where lower ranks are preferred). The performance comparison shows that the most accurate estimation method is mostly dependent on the sample size (n). At the lowest sample size ($n = 15$), the MSADE and MSALDE estimators have the highest average performance, standing at first and second positions, respectively, followed by the third position occupied by the MLE. As sample sizes are raised to moderate ($n = 50$ and $n = 120$), the MLE and MSADE estimators trade places, confirming both to be high achievers and viable methods for this interval. A steep decline at $n = 180$ occurs when the LTADE holds the lowest rank sum, although the MLE returns to the top yet again at $n = 250$. For the largest sample size ($n = 350$), the MLE achieves its expected asymptotic efficiency, having a highly dominating first rank with an extremely low sum of ranks (12.0[1]). The WLSE has the second-best performance in this case. Conversely, KE and estimators like MSLNDE and MSSDE always do the worst overall, typically performing last in every sample size, which suggests they are not suitable to apply when estimating the parameters of this model. In summary, MLE is asymptotically most efficient, and MSADE and MSALDE are more desirable for small-to-moderate sample sizes. Figures 5, 7, 6 and 8 are plots for the Bias, MSE, MRE, D_{abs} and D_{max} which support the results in Table III.

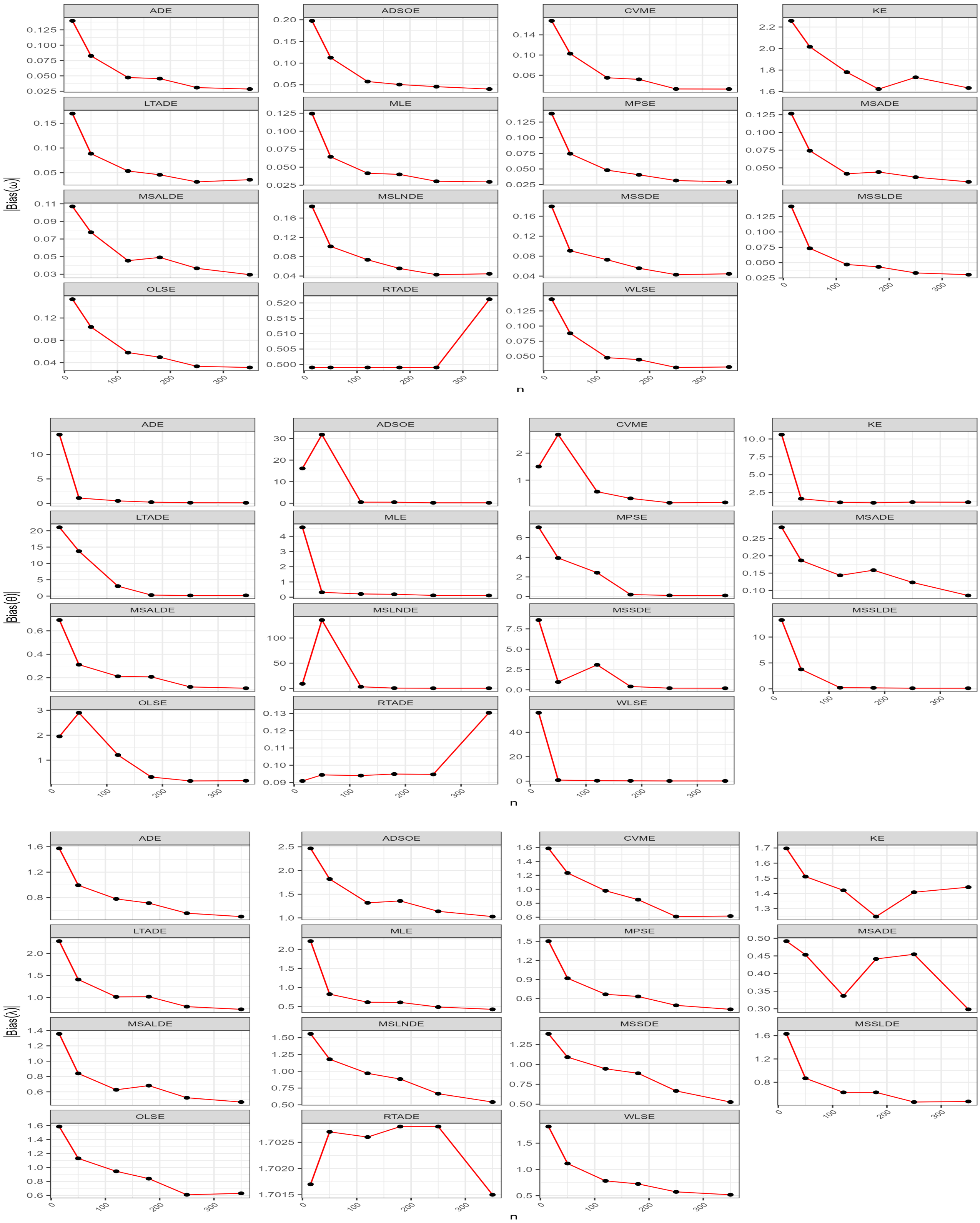

Fig. 9. Graphical representations for the Bias values presented in Table V.

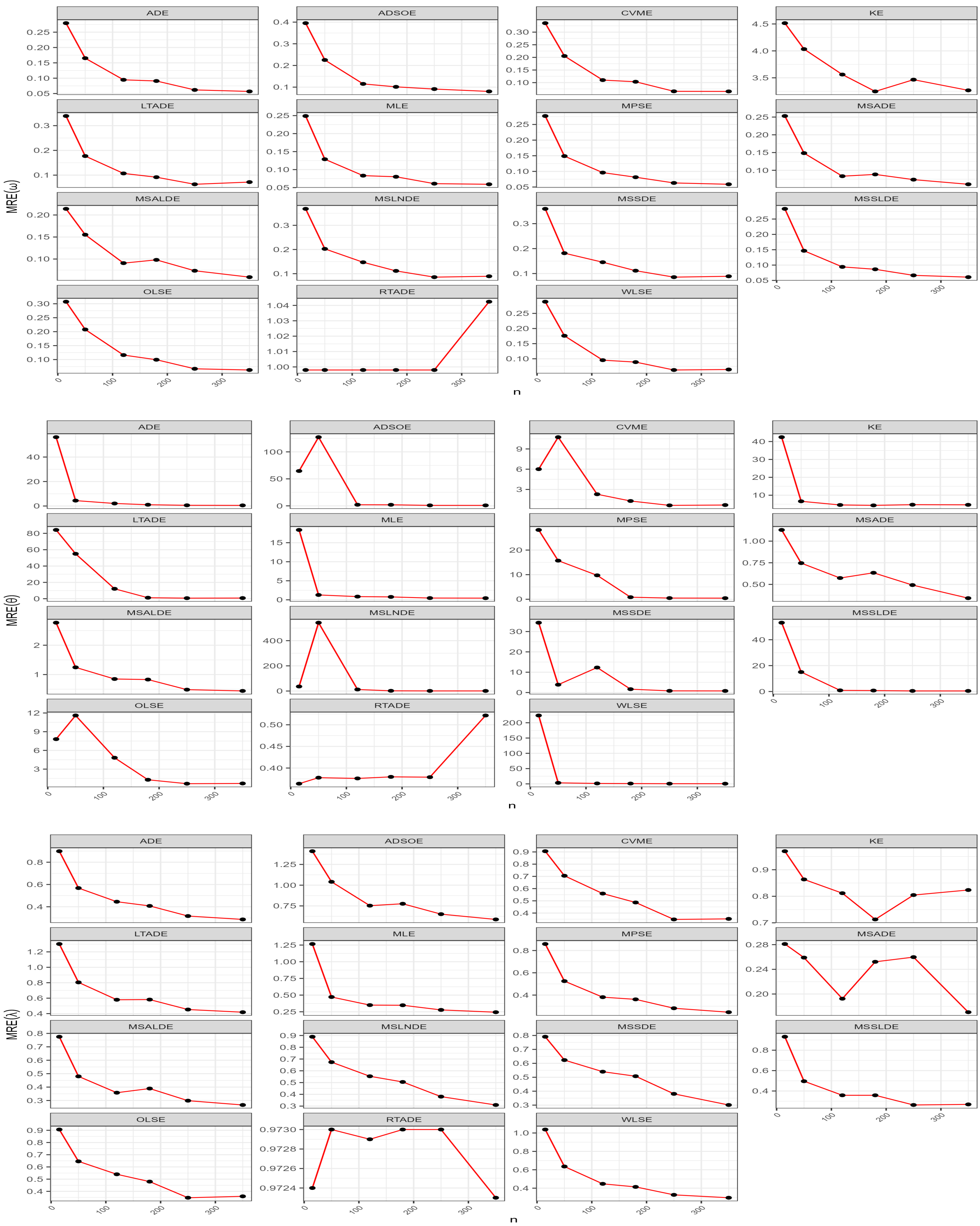

Fig. 10. Graphical representations for the MRE values presented in Table V.

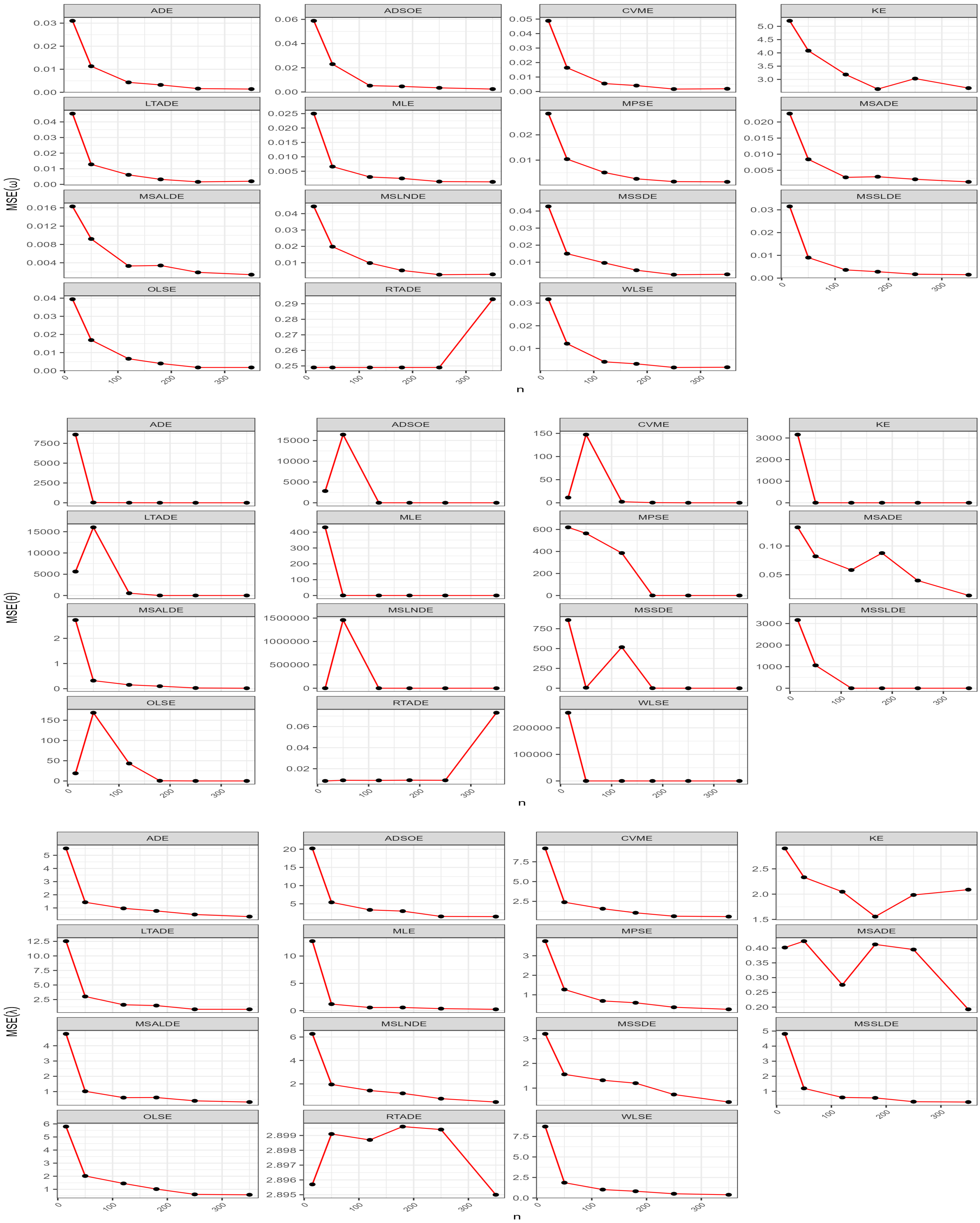

Fig. 11. Graphical representations for the MSE values presented in Table V.

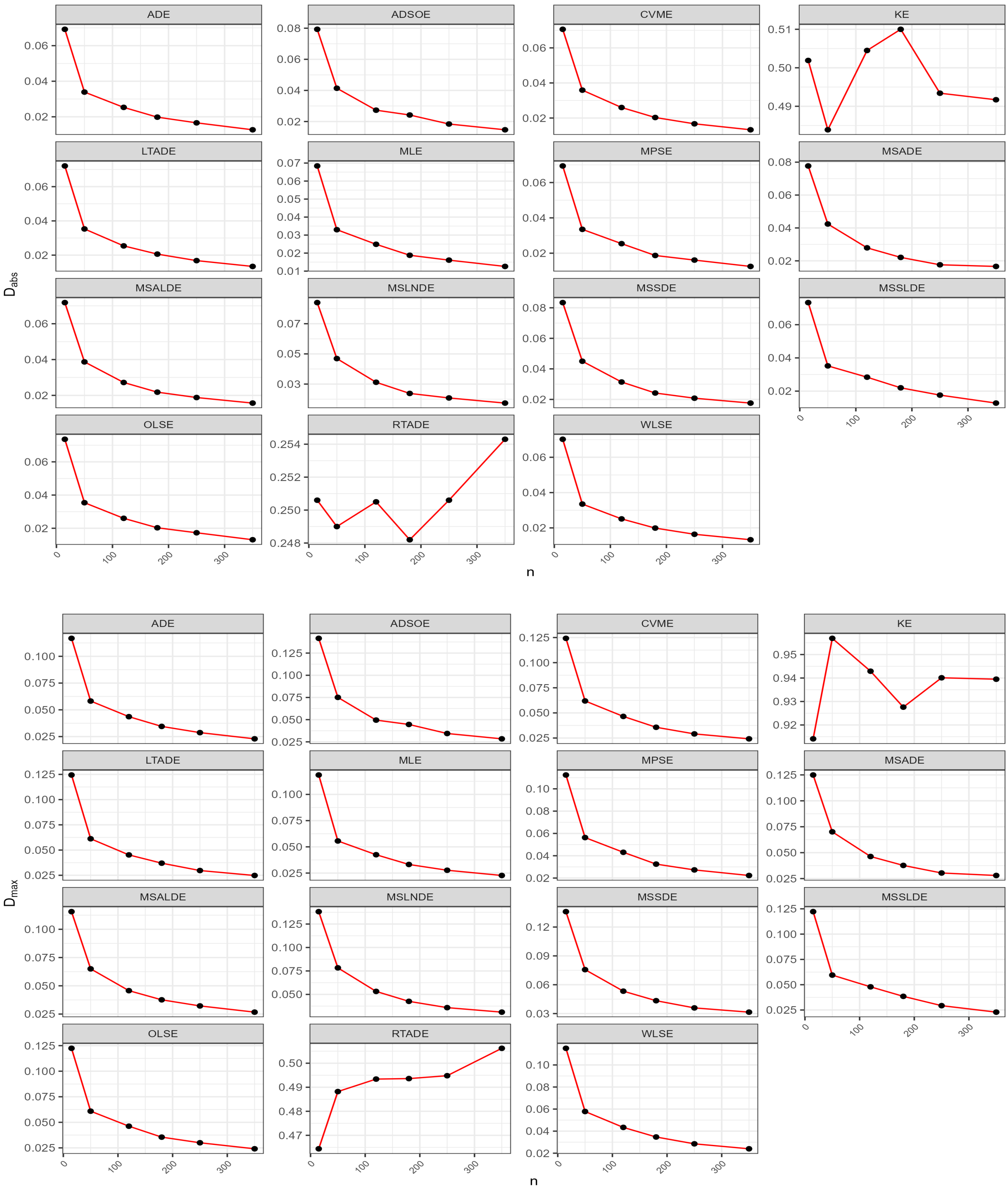

Fig. 12. Graphical representations for the D_{abs} and D_{max} values presented in Table V.

TABLE V. Numerical values of simulation measures for $\theta = 0.25,\ \lambda = 1.75\ \omega = 0.5$.

n	Metric	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
15	$\lvert Bias\rvert(\hat{\theta})$	$4.589^{\{6\}}$	$14.06^{\{12\}}$	$1.502^{\{4\}}$	$7.048^{\{7\}}$	$1.952^{\{5\}}$	$0.0909^{\{1\}}$	$55.95^{\{15\}}$	$21.05^{\{14\}}$	$0.2823^{\{2\}}$	$0.6918^{\{3\}}$	$16.14^{\{13\}}$	$10.61^{\{10\}}$	$8.587^{\{8\}}$	$13.28^{\{11\}}$	$8.969^{\{9\}}$
	$\lvert Bias\rvert(\hat{\lambda})$	$2.216^{\{13\}}$	$1.573^{\{6\}}$	$1.585^{\{7\}}$	$1.503^{\{4\}}$	$1.586^{\{8\}}$	$1.702^{\{11\}}$	$1.816^{\{12\}}$	$2.279^{\{14\}}$	$0.4919^{\{1\}}$	$1.356^{\{2\}}$	$2.466^{\{15\}}$	$1.697^{\{10\}}$	$1.384^{\{3\}}$	$1.631^{\{9\}}$	$1.556^{\{5\}}$
	$\lvert Bias\rvert(\hat{\omega})$	$0.1244^{\{2\}}$	$0.1397^{\{5\}}$	$0.1678^{\{9\}}$	$0.1385^{\{4\}}$	$0.1536^{\{8\}}$	$0.499^{\{14\}}$	$0.1443^{\{7\}}$	$0.1695^{\{10\}}$	$0.1265^{\{3\}}$	$0.107^{\{1\}}$	$0.1976^{\{13\}}$	$2.258^{\{15\}}$	$0.1796^{\{11\}}$	$0.1416^{\{6\}}$	$0.184^{\{12\}}$
	MSE ($\hat{\theta}$)	$430.3^{\{6\}}$	$8598^{\{14\}}$	$11.35^{\{4\}}$	$619.2^{\{7\}}$	$18.9^{\{5\}}$	$0.0084^{\{1\}}$	$2.567e+05^{\{15\}}$	$5628^{\{13\}}$	$0.1325^{\{2\}}$	$2.728^{\{3\}}$	$2832^{\{10\}}$	$3157^{\{11\}}$	$859.1^{\{9\}}$	$3162^{\{12\}}$	$850.2^{\{8\}}$
	MSE ($\hat{\lambda}$)	$12.72^{\{14\}}$	$5.522^{\{8\}}$	$9.187^{\{12\}}$	$3.748^{\{5\}}$	$5.795^{\{9\}}$	$2.896^{\{2\}}$	$8.703^{\{11\}}$	$12.54^{\{13\}}$	$0.4019^{\{1\}}$	$4.776^{\{6\}}$	$20.22^{\{15\}}$	$2.904^{\{3\}}$	$3.196^{\{4\}}$	$4.813^{\{7\}}$	$6.268^{\{10\}}$
	MSE ($\hat{\omega}$)	$0.025^{\{3\}}$	$0.031^{\{5\}}$	$0.0488^{\{12\}}$	$0.0284^{\{4\}}$	$0.0394^{\{8\}}$	$0.249^{\{14\}}$	$0.0317^{\{7\}}$	$0.0453^{\{11\}}$	$0.0226^{\{2\}}$	$0.0163^{\{1\}}$	$0.0588^{\{13\}}$	$5.208^{\{15\}}$	$0.0427^{\{9\}}$	$0.0315^{\{6\}}$	$0.0444^{\{10\}}$
	MRE ($\hat{\theta}$)	$18.36^{\{6\}}$	$56.23^{\{12\}}$	$6.006^{\{4\}}$	$28.19^{\{7\}}$	$7.809^{\{5\}}$	$0.3638^{\{1\}}$	$223.8^{\{15\}}$	$84.19^{\{14\}}$	$1.129^{\{2\}}$	$2.767^{\{3\}}$	$64.56^{\{13\}}$	$42.44^{\{10\}}$	$34.35^{\{8\}}$	$53.13^{\{11\}}$	$35.88^{\{9\}}$
	MRE ($\hat{\lambda}$)	$1.266^{\{13\}}$	$0.8987^{\{6\}}$	$0.9058^{\{7\}}$	$0.8587^{\{4\}}$	$0.9063^{\{8\}}$	$0.9724^{\{11\}}$	$1.038^{\{12\}}$	$1.302^{\{14\}}$	$0.2811^{\{1\}}$	$0.775^{\{2\}}$	$1.409^{\{15\}}$	$0.9698^{\{10\}}$	$0.7908^{\{3\}}$	$0.9317^{\{9\}}$	$0.8893^{\{5\}}$
	MRE ($\hat{\omega}$)	$0.2489^{\{2\}}$	$0.2794^{\{5\}}$	$0.3356^{\{9\}}$	$0.2771^{\{4\}}$	$0.3073^{\{8\}}$	$0.998^{\{14\}}$	$0.2886^{\{7\}}$	$0.339^{\{10\}}$	$0.253^{\{3\}}$	$0.2139^{\{1\}}$	$0.3953^{\{13\}}$	$4.515^{\{15\}}$	$0.3593^{\{11\}}$	$0.2832^{\{6\}}$	$0.3681^{\{12\}}$
	D_{abs}	$0.0684^{\{1\}}$	$0.0692^{\{2\}}$	$0.0706^{\{5\}}$	$0.0694^{\{3\}}$	$0.0736^{\{9\}}$	$0.2506^{\{14\}}$	$0.0702^{\{4\}}$	$0.0721^{\{7\}}$	$0.0777^{\{10\}}$	$0.0718^{\{6\}}$	$0.0793^{\{11\}}$	$0.5019^{\{15\}}$	$0.0834^{\{12\}}$	$0.0733^{\{8\}}$	$0.084^{\{13\}}$
	D_{max}	$0.1185^{\{5\}}$	$0.1168^{\{4\}}$	$0.1242^{\{8\}}$	$0.1124^{\{1\}}$	$0.1224^{\{7\}}$	$0.4644^{\{14\}}$	$0.1151^{\{2\}}$	$0.1244^{\{9\}}$	$0.1249^{\{10\}}$	$0.1156^{\{3\}}$	$0.1416^{\{13\}}$	$0.9141^{\{15\}}$	$0.136^{\{11\}}$	$0.1222^{\{6\}}$	$0.1382^{\{12\}}$
	$\sum$Ranks	$71^{\{5\}}$	$79^{\{6\}}$	$81^{\{8\}}$	$50^{\{3\}}$	$80^{\{7\}}$	$97^{\{11\}}$	$107^{\{13\}}$	$129^{\{13.5\}}$	$37^{\{1\}}$	$31^{\{2\}}$	$144^{\{15\}}$	$129^{\{13.5\}}$	$89^{\{9\}}$	$91^{\{10\}}$	$105^{\{12\}}$
50	$\lvert Bias\rvert(\hat{\theta})$	$0.3195^{\{4\}}$	$1.102^{\{7\}}$	$2.686^{\{9\}}$	$3.925^{\{12\}}$	$2.9^{\{10\}}$	$0.0944^{\{1\}}$	$0.844^{\{5\}}$	$13.74^{\{13\}}$	$0.1865^{\{2\}}$	$0.3113^{\{3\}}$	$31.81^{\{14\}}$	$1.651^{\{8\}}$	$0.9628^{\{6\}}$	$3.759^{\{11\}}$	$135.7^{\{15\}}$
	$\lvert Bias\rvert(\hat{\lambda})$	$0.8249^{\{2\}}$	$0.9939^{\{6\}}$	$1.234^{\{11\}}$	$0.9193^{\{5\}}$	$1.13^{\{9\}}$	$1.703^{\{14\}}$	$1.112^{\{8\}}$	$1.408^{\{12\}}$	$0.4532^{\{1\}}$	$0.8397^{\{3\}}$	$1.821^{\{15\}}$	$1.511^{\{13\}}$	$1.091^{\{7\}}$	$0.8679^{\{4\}}$	$1.179^{\{10\}}$
	$\lvert Bias\rvert(\hat{\omega})$	$0.0644^{\{1\}}$	$0.0826^{\{6\}}$	$0.1027^{\{11\}}$	$0.0744^{\{4\}}$	$0.1039^{\{12\}}$	$0.499^{\{14\}}$	$0.088^{\{7\}}$	$0.0885^{\{8\}}$	$0.0742^{\{3\}}$	$0.0776^{\{5\}}$	$0.1126^{\{13\}}$	$2.017^{\{15\}}$	$0.0908^{\{9\}}$	$0.0733^{\{2\}}$	$0.1013^{\{10\}}$
	MSE ($\hat{\theta}$)	$0.4016^{\{4\}}$	$33.97^{\{8\}}$	$147.4^{\{9\}}$	$564.2^{\{11\}}$	$168.5^{\{10\}}$	$0.009^{\{1\}}$	$5.093^{\{6\}}$	$1.603e+04^{\{13\}}$	$0.082^{\{2\}}$	$0.3197^{\{3\}}$	$1.641e+04^{\{14\}}$	$3.842^{\{5\}}$	$6.702^{\{7\}}$	$1060^{\{12\}}$	$1.458e+06^{\{15\}}$
	MSE ($\hat{\lambda}$)	$1.203^{\{4\}}$	$1.431^{\{6\}}$	$2.393^{\{12\}}$	$1.272^{\{5\}}$	$2.018^{\{10\}}$	$2.899^{\{13\}}$	$1.865^{\{8\}}$	$3.025^{\{14\}}$	$0.4236^{\{1\}}$	$1.02^{\{2\}}$	$5.413^{\{15\}}$	$2.333^{\{11\}}$	$1.557^{\{7\}}$	$1.196^{\{3\}}$	$1.953^{\{9\}}$
	MSE ($\hat{\omega}$)	$0.0066^{\{1\}}$	$0.0113^{\{6\}}$	$0.0164^{\{10\}}$	$0.0104^{\{5\}}$	$0.0169^{\{11\}}$	$0.249^{\{14\}}$	$0.0121^{\{7\}}$	$0.0128^{\{8\}}$	$0.0084^{\{2\}}$	$0.0092^{\{4\}}$	$0.023^{\{13\}}$	$4.082^{\{15\}}$	$0.015^{\{9\}}$	$0.009^{\{3\}}$	$0.0198^{\{12\}}$
	MRE ($\hat{\theta}$)	$1.278^{\{4\}}$	$4.409^{\{7\}}$	$10.74^{\{9\}}$	$15.7^{\{12\}}$	$11.6^{\{10\}}$	$0.3777^{\{1\}}$	$3.376^{\{5\}}$	$54.96^{\{13\}}$	$0.746^{\{2\}}$	$1.245^{\{3\}}$	$127.2^{\{14\}}$	$6.606^{\{8\}}$	$3.851^{\{6\}}$	$15.03^{\{11\}}$	$542.7^{\{15\}}$
	MRE ($\hat{\lambda}$)	$0.4714^{\{2\}}$	$0.5679^{\{6\}}$	$0.705^{\{11\}}$	$0.5253^{\{5\}}$	$0.6456^{\{9\}}$	$0.973^{\{14\}}$	$0.6356^{\{8\}}$	$0.8047^{\{12\}}$	$0.259^{\{1\}}$	$0.4798^{\{3\}}$	$1.04^{\{15\}}$	$0.8635^{\{13\}}$	$0.6234^{\{7\}}$	$0.4959^{\{4\}}$	$0.6736^{\{10\}}$
	MRE ($\hat{\omega}$)	$0.1288^{\{1\}}$	$0.1652^{\{6\}}$	$0.2054^{\{11\}}$	$0.1489^{\{4\}}$	$0.2078^{\{12\}}$	$0.998^{\{14\}}$	$0.176^{\{7\}}$	$0.1771^{\{8\}}$	$0.1484^{\{3\}}$	$0.1552^{\{5\}}$	$0.2253^{\{13\}}$	$4.033^{\{15\}}$	$0.1816^{\{9\}}$	$0.1466^{\{2\}}$	$0.2026^{\{10\}}$
	D_{abs}	$0.033^{\{1\}}$	$0.0339^{\{4\}}$	$0.0359^{\{8\}}$	$0.0335^{\{2\}}$	$0.0354^{\{7\}}$	$0.249^{\{14\}}$	$0.0335^{\{3\}}$	$0.0353^{\{6\}}$	$0.0424^{\{11\}}$	$0.0387^{\{9\}}$	$0.0414^{\{10\}}$	$0.4839^{\{15\}}$	$0.045^{\{12\}}$	$0.0352^{\{5\}}$	$0.0469^{\{13\}}$
	D_{max}	$0.0556^{\{1\}}$	$0.0582^{\{4\}}$	$0.0619^{\{8\}}$	$0.0563^{\{2\}}$	$0.0609^{\{6\}}$	$0.4882^{\{14\}}$	$0.0578^{\{3\}}$	$0.0612^{\{7\}}$	$0.0701^{\{10\}}$	$0.065^{\{9\}}$	$0.0751^{\{11\}}$	$0.9569^{\{15\}}$	$0.0757^{\{12\}}$	$0.0595^{\{5\}}$	$0.0783^{\{13\}}$
	$\sum$Ranks	$25^{\{1\}}$	$66^{\{5\}}$	$109^{\{10\}}$	$67^{\{6.5\}}$	$106^{\{9\}}$	$114^{\{11.5\}}$	$67^{\{6.5\}}$	$114^{\{11.5\}}$	$38^{\{2\}}$	$49^{\{3\}}$	$147^{\{15\}}$	$133^{\{14\}}$	$91^{\{8\}}$	$62^{\{4\}}$	$132^{\{13\}}$
120	$\lvert Bias\rvert(\hat{\theta})$	$0.2109^{\{3\}}$	$0.5182^{\{8\}}$	$0.5721^{\{9\}}$	$2.432^{\{12\}}$	$1.206^{\{11\}}$	$0.094^{\{1\}}$	$0.3505^{\{6\}}$	$3.035^{\{13\}}$	$0.1434^{\{2\}}$	$0.212^{\{4\}}$	$0.5121^{\{7\}}$	$1.137^{\{10\}}$	$3.075^{\{15\}}$	$0.223^{\{5\}}$	$3.059^{\{14\}}$
	$\lvert Bias\rvert(\hat{\lambda})$	$0.6125^{\{2\}}$	$0.7793^{\{6\}}$	$0.9787^{\{11\}}$	$0.6671^{\{5\}}$	$0.9447^{\{8\}}$	$1.703^{\{15\}}$	$0.7824^{\{7\}}$	$1.014^{\{12\}}$	$0.3366^{\{1\}}$	$0.6267^{\{4\}}$	$1.318^{\{13\}}$	$1.42^{\{14\}}$	$0.9449^{\{9\}}$	$0.6265^{\{3\}}$	$0.9686^{\{10\}}$
	$\lvert Bias\rvert(\hat{\omega})$	$0.0416^{\{1\}}$	$0.0473^{\{5\}}$	$0.055^{\{9\}}$	$0.0481^{\{7\}}$	$0.058^{\{11\}}$	$0.499^{\{14\}}$	$0.0477^{\{6\}}$	$0.0535^{\{8\}}$	$0.0417^{\{2\}}$	$0.0454^{\{3\}}$	$0.0575^{\{10\}}$	$1.78^{\{15\}}$	$0.0727^{\{12\}}$	$0.047^{\{4\}}$	$0.0736^{\{13\}}$
	MSE ($\hat{\theta}$)	$0.1307^{\{3\}}$	$5.915^{\{10\}}$	$2.321^{\{9\}}$	$385.7^{\{12\}}$	$43.15^{\{11\}}$	$0.0089^{\{1\}}$	$0.793^{\{6\}}$	$543.1^{\{15\}}$	$0.0582^{\{2\}}$	$0.1523^{\{5\}}$	$1.071^{\{7\}}$	$1.294^{\{8\}}$	$517.2^{\{14\}}$	$0.1437^{\{4\}}$	$500.2^{\{13\}}$
	MSE ($\hat{\lambda}$)	$0.5762^{\{2\}}$	$0.9668^{\{6\}}$	$1.57^{\{11\}}$	$0.6867^{\{5\}}$	$1.447^{\{10\}}$	$2.899^{\{14\}}$	$1.022^{\{7\}}$	$1.601^{\{12\}}$	$0.2755^{\{1\}}$	$0.6039^{\{4\}}$	$3.336^{\{15\}}$	$2.047^{\{13\}}$	$1.318^{\{8\}}$	$0.5909^{\{3\}}$	$1.435^{\{9\}}$
	MSE ($\hat{\omega}$)	$0.003^{\{2\}}$	$0.0043^{\{6\}}$	$0.0055^{\{9\}}$	$0.0051^{\{7\}}$	$0.0066^{\{11\}}$	$0.249^{\{14\}}$	$0.0041^{\{5\}}$	$0.0061^{\{10\}}$	$0.0028^{\{1\}}$	$0.0033^{\{3\}}$	$0.0052^{\{8\}}$	$3.184^{\{15\}}$	$0.0096^{\{12\}}$	$0.0036^{\{4\}}$	$0.0098^{\{13\}}$
	MRE ($\hat{\theta}$)	$0.8435^{\{3\}}$	$2.073^{\{8\}}$	$2.288^{\{9\}}$	$9.727^{\{12\}}$	$4.823^{\{11\}}$	$0.376^{\{1\}}$	$1.402^{\{6\}}$	$12.14^{\{13\}}$	$0.5737^{\{2\}}$	$0.8482^{\{4\}}$	$2.049^{\{7\}}$	$4.548^{\{10\}}$	$12.3^{\{15\}}$	$0.8918^{\{5\}}$	$12.23^{\{14\}}$
	MRE ($\hat{\lambda}$)	$0.35^{\{2\}}$	$0.4453^{\{6\}}$	$0.5593^{\{11\}}$	$0.3812^{\{5\}}$	$0.5398^{\{8\}}$	$0.9729^{\{15\}}$	$0.4471^{\{7\}}$	$0.5794^{\{12\}}$	$0.1924^{\{1\}}$	$0.3581^{\{4\}}$	$0.753^{\{13\}}$	$0.8116^{\{14\}}$	$0.54^{\{9\}}$	$0.358^{\{3\}}$	$0.5535^{\{10\}}$
	MRE ($\hat{\omega}$)	$0.0832^{\{1\}}$	$0.0947^{\{5\}}$	$0.1101^{\{9\}}$	$0.0961^{\{7\}}$	$0.1161^{\{11\}}$	$0.998^{\{14\}}$	$0.0954^{\{6\}}$	$0.1069^{\{8\}}$	$0.0834^{\{2\}}$	$0.0907^{\{3\}}$	$0.1149^{\{10\}}$	$3.561^{\{15\}}$	$0.1454^{\{12\}}$	$0.0941^{\{4\}}$	$0.1472^{\{13\}}$
	D_{abs}	$0.0249^{\{1\}}$	$0.0253^{\{3\}}$	$0.026^{\{7\}}$	$0.0254^{\{4\}}$	$0.026^{\{6\}}$	$0.2505^{\{14\}}$	$0.0251^{\{2\}}$	$0.0254^{\{5\}}$	$0.0279^{\{10\}}$	$0.0272^{\{8\}}$	$0.0273^{\{9\}}$	$0.5045^{\{15\}}$	$0.0314^{\{13\}}$	$0.0284^{\{11\}}$	$0.0312^{\{12\}}$
	D_{max}	$0.0425^{\{1\}}$	$0.0436^{\{4\}}$	$0.0465^{\{9\}}$	$0.0431^{\{2\}}$	$0.0462^{\{7\}}$	$0.4934^{\{14\}}$	$0.0434^{\{3\}}$	$0.0452^{\{5\}}$	$0.0463^{\{8\}}$	$0.0457^{\{6\}}$	$0.0495^{\{11\}}$	$0.9429^{\{15\}}$	$0.0533^{\{13\}}$	$0.0479^{\{10\}}$	$0.0532^{\{12\}}$
	$\sum$Ranks	$21^{\{1\}}$	$67^{\{6\}}$	$103^{\{8\}}$	$78^{\{7\}}$	$105^{\{9\}}$	$117^{\{13\}}$	$61^{\{5\}}$	$113^{\{11.5\}}$	$32^{\{2\}}$	$48^{\{3\}}$	$110^{\{10\}}$	$144^{\{14\}}$	$132^{\{15\}}$	$56^{\{4\}}$	$133^{\{11.5\}}$
180	$\lvert Bias\rvert(\hat{\theta})$	$0.1902^{\{4\}}$	$0.2585^{\{8\}}$	$0.3225^{\{11\}}$	$0.2081^{\{6\}}$	$0.3225^{\{10\}}$	$0.0949^{\{1\}}$	$0.257^{\{7\}}$	$0.3044^{\{9\}}$	$0.1585^{\{2\}}$	$0.2074^{\{5\}}$	$0.4895^{\{14\}}$	$1.08^{\{15\}}$	$0.4121^{\{12\}}$	$0.188^{\{3\}}$	$0.4155^{\{13\}}$
	$\lvert Bias\rvert(\hat{\lambda})$	$0.6087^{\{2\}}$	$0.7133^{\{6\}}$	$0.8514^{\{9\}}$	$0.6328^{\{4\}}$	$0.8395^{\{8\}}$	$1.703^{\{15\}}$	$0.7242^{\{7\}}$	$1.018^{\{12\}}$	$0.4414^{\{1\}}$	$0.6811^{\{5\}}$	$1.358^{\{14\}}$	$1.247^{\{13\}}$	$0.8881^{\{11\}}$	$0.6265^{\{3\}}$	$0.884^{\{10\}}$
	$\lvert Bias\rvert(\hat{\omega})$	$0.04^{\{1\}}$	$0.0455^{\{6\}}$	$0.0518^{\{11\}}$	$0.0407^{\{2\}}$	$0.0497^{\{9\}}$	$0.499^{\{14\}}$	$0.0445^{\{5\}}$	$0.0458^{\{7\}}$	$0.0442^{\{4\}}$	$0.0491^{\{8\}}$	$0.0506^{\{10\}}$	$1.624^{\{15\}}$	$0.0557^{\{12\}}$	$0.0431^{\{3\}}$	$0.0557^{\{13\}}$
	MSE ($\hat{\theta}$)	$0.119^{\{5\}}$	$0.2757^{\{8\}}$	$0.427^{\{10\}}$	$0.1628^{\{6\}}$	$0.486^{\{11\}}$	$0.0091^{\{1\}}$	$0.2638^{\{7\}}$	$0.2791^{\{9\}}$	$0.0877^{\{2\}}$	$0.099^{\{4\}}$	$1.781^{\{15\}}$	$1.167^{\{14\}}$	$1.065^{\{12\}}$	$0.0896^{\{3\}}$	$1.081^{\{13\}}$
	MSE ($\hat{\lambda}$)	$0.582^{\{3\}}$	$0.7719^{\{6\}}$	$1.056^{\{9\}}$	$0.5935^{\{4\}}$	$1.017^{\{8\}}$	$2.9^{\{14\}}$	$0.8203^{\{7\}}$	$1.467^{\{12\}}$	$0.4125^{\{1\}}$	$0.6113^{\{5\}}$	$3.001^{\{15\}}$	$1.556^{\{13\}}$	$1.2^{\{11\}}$	$0.5614^{\{2\}}$	$1.195^{\{10\}}$
	MSE ($\hat{\omega}$)	$0.0025^{\{1\}}$	$0.0032^{\{7\}}$	$0.0041^{\{10\}}$	$0.0026^{\{2\}}$	$0.004^{\{9\}}$	$0.249^{\{14\}}$	$0.0032^{\{5\}}$	$0.0032^{\{6\}}$	$0.003^{\{4\}}$	$0.0034^{\{8\}}$	$0.0046^{\{11\}}$	$2.636^{\{15\}}$	$0.0053^{\{12\}}$	$0.0028^{\{3\}}$	$0.0053^{\{13\}}$
	MRE ($\hat{\theta}$)	$0.7606^{\{4\}}$	$1.034^{\{8\}}$	$1.29^{\{11\}}$	$0.8325^{\{6\}}$	$1.29^{\{10\}}$	$0.3796^{\{1\}}$	$1.028^{\{7\}}$	$1.218^{\{9\}}$	$0.6341^{\{2\}}$	$0.8296^{\{5\}}$	$1.958^{\{14\}}$	$4.321^{\{15\}}$	$1.649^{\{12\}}$	$0.7519^{\{3\}}$	$1.662^{\{13\}}$
	MRE ($\hat{\lambda}$)	$0.3478^{\{2\}}$	$0.4076^{\{6\}}$	$0.4865^{\{9\}}$	$0.3616^{\{4\}}$	$0.4797^{\{8\}}$	$0.973^{\{15\}}$	$0.4139^{\{7\}}$	$0.5817^{\{12\}}$	$0.2522^{\{1\}}$	$0.3892^{\{5\}}$	$0.776^{\{14\}}$	$0.7127^{\{13\}}$	$0.5075^{\{11\}}$	$0.358^{\{3\}}$	$0.5052^{\{10\}}$
	MRE ($\hat{\omega}$)	$0.0801^{\{1\}}$	$0.0909^{\{6\}}$	$0.1035^{\{11\}}$	$0.0814^{\{2\}}$	$0.0994^{\{9\}}$	$0.998^{\{14\}}$	$0.089^{\{5\}}$	$0.0916^{\{7\}}$	$0.0884^{\{4\}}$	$0.0981^{\{8\}}$	$0.1013^{\{10\}}$	$3.247^{\{15\}}$	$0.1114^{\{12\}}$	$0.0863^{\{3\}}$	$0.1115^{\{13\}}$
	D_{abs}	$0.0188^{\{2\}}$	$0.0198^{\{3\}}$	$0.0203^{\{5\}}$	$0.0187^{\{1\}}$	$0.0203^{\{6\}}$	$0.2482^{\{14\}}$	$0.0199^{\{4\}}$	$0.0206^{\{7\}}$	$0.0221^{\{10\}}$	$0.0218^{\{8\}}$	$0.0242^{\{12\}}$	$0.51^{\{15\}}$	$0.0242^{\{13\}}$	$0.022^{\{9\}}$	$0.0238^{\{11\}}$
	D_{max}	$0.0332^{\{2\}}$	$0.0345^{\{3\}}$	$0.0357^{\{6\}}$	$0.0325^{\{1\}}$	$0.0355^{\{5\}}$	$0.4936^{\{14\}}$	$0.0347^{\{4\}}$	$0.037^{\{7\}}$	$0.0377^{\{9\}}$	$0.0376^{\{8\}}$	$0.0446^{\{13\}}$	$0.9276^{\{15\}}$	$0.0434^{\{12\}}$	$0.0385^{\{10\}}$	$0.0426^{\{11\}}$
	$\sum$Ranks	$27^{\{1\}}$	$67^{\{6\}}$	$102^{\{10\}}$	$38^{\{2\}}$	$93^{\{8\}}$	$117^{\{11\}}$	$65^{\{5\}}$	$97^{\{9\}}$	$40^{\{3\}}$	$69^{\{7\}}$	$142^{\{14\}}$	$158^{\{15\}}$	$130^{\{12.5\}}$	$45^{\{4\}}$	$130^{\{12.5\}}$
250	$\lvert Bias\rvert(\hat{\theta})$	$0.1155^{\{2\}}$	$0.1439^{\{7\}}$	$0.1602^{\{9\}}$	$0.1192^{\{4\}}$	$0.1656^{\{10\}}$	$0.0947^{\{1\}}$	$0.1463^{\{8\}}$	$0.1746^{\{11\}}$	$0.1231^{\{6\}}$	$0.1213^{\{5\}}$	$0.2203^{\{14\}}$	$1.174^{\{15\}}$	$0.2061^{\{12\}}$	$0.1175^{\{3\}}$	$0.2064^{\{13\}}$
	$\lvert Bias\rvert(\hat{\lambda})$	$0.4863^{\{3\}}$	$0.5544^{\{6\}}$	$0.6087^{\{9\}}$	$0.493^{\{4\}}$	$0.6083^{\{8\}}$	$1.703^{\{15\}}$	$0.573^{\{7\}}$	$0.7922^{\{12\}}$	$0.4547^{\{1\}}$	$0.5227^{\{5\}}$	$1.139^{\{13\}}$	$1.408^{\{14\}}$	$0.6664^{\{11\}}$	$0.4603^{\{2\}}$	$0.6658^{\{10\}}$
	$\lvert Bias\rvert(\hat{\omega})$	$0.0304^{\{1\}}$	$0.0309^{\{2\}}$	$0.0328^{\{6\}}$	$0.0315^{\{5\}}$	$0.0334^{\{8\}}$	$0.499^{\{14\}}$	$0.0315^{\{3\}}$	$0.0315^{\{4\}}$	$0.0369^{\{10\}}$	$0.0367^{\{9\}}$	$0.0456^{\{13\}}$	$1.733^{\{15\}}$	$0.0428^{\{11\}}$	$0.0332^{\{7\}}$	$0.0428^{\{12\}}$
	MSE ($\hat{\theta}$)	$0.0264^{\{2\}}$	$0.046^{\{7\}}$	$0.0635^{\{10\}}$	$0.0273^{\{3\}}$	$0.0669^{\{11\}}$	$0.009^{\{1\}}$	$0.0475^{\{8\}}$	$0.0532^{\{9\}}$	$0.0399^{\{6\}}$	$0.0275^{\{4\}}$	$0.0675^{\{12\}}$	$1.379^{\{15\}}$	$0.0973^{\{13\}}$	$0.034^{\{5\}}$	$0.0977^{\{14\}}$
	MSE ($\hat{\lambda}$)	$0.3747^{\{3\}}$	$0.4989^{\{6\}}$	$0.6241^{\{9\}}$	$0.3629^{\{2\}}$	$0.6106^{\{8\}}$	$2.899^{\{15\}}$	$0.5117^{\{7\}}$	$0.8234^{\{12\}}$	$0.3951^{\{5\}}$	$0.3946^{\{4\}}$	$1.534^{\{13\}}$	$1.984^{\{14\}}$	$0.7406^{\{10\}}$	$0.3086^{\{1\}}$	$0.7418^{\{11\}}$
	MSE ($\hat{\omega}$)	$0.0014^{\{1\}}$	$0.0016^{\{3\}}$	$0.0017^{\{7\}}$	$0.0015^{\{2\}}$	$0.0018^{\{8\}}$	$0.249^{\{14\}}$	$0.0016^{\{5\}}$	$0.0016^{\{4\}}$	$0.0022^{\{10\}}$	$0.0019^{\{9\}}$	$0.0034^{\{13\}}$	$3.031^{\{15\}}$	$0.0027^{\{11\}}$	$0.0017^{\{6\}}$	$0.0027^{\{12\}}$
	MRE ($\hat{\theta}$)	$0.4619^{\{2\}}$	$0.5756^{\{7\}}$	$0.6408^{\{9\}}$	$0.4767^{\{4\}}$	$0.6623^{\{10\}}$	$0.3789^{\{1\}}$	$0.5853^{\{8\}}$	$0.6982^{\{11\}}$	$0.4925^{\{6\}}$	$0.4852^{\{5\}}$	$0.8812^{\{14\}}$	$4.696^{\{15\}}$	$0.8243^{\{12\}}$	$0.4699^{\{3\}}$	$0.8257^{\{13\}}$
	MRE ($\hat{\lambda}$)	$0.2779^{\{3\}}$	$0.3168^{\{6\}}$	$0.3478^{\{9\}}$	$0.2817^{\{4\}}$	$0.3476^{\{8\}}$	$0.973^{\{15\}}$	$0.3274^{\{7\}}$	$0.4527^{\{12\}}$	$0.2598^{\{1\}}$	$0.2987^{\{5\}}$	$0.6507^{\{13\}}$	$0.8046^{\{14\}}$	$0.3808^{\{11\}}$	$0.263^{\{2\}}$	$0.3804^{\{10\}}$
	MRE ($\hat{\omega}$)	$0.0609^{\{1\}}$	$0.0618^{\{2\}}$	$0.0656^{\{6\}}$	$0.0631^{\{5\}}$	$0.0669^{\{8\}}$	$0.998^{\{14\}}$	$0.0629^{\{3\}}$	$0.063^{\{4\}}$	$0.0737^{\{10\}}$	$0.0734^{\{9\}}$	$0.0911^{\{13\}}$	$3.466^{\{15\}}$	$0.0856^{\{11\}}$	$0.0665^{\{7\}}$	$0.0857^{\{12\}}$
	D_{abs}	$0.0161^{\{2\}}$	$0.0166^{\{4\}}$	$0.0167^{\{5\}}$	$0.0161^{\{1\}}$	$0.0173^{\{7\}}$	$0.2506^{\{14\}}$	$0.0164^{\{3\}}$	$0.0168^{\{6\}}$	$0.0176^{\{8\}}$	$0.0188^{\{11\}}$	$0.0184^{\{10\}}$	$0.4934^{\{15\}}$	$0.0208^{\{12\}}$	$0.0176^{\{9\}}$	$0.0208^{\{13\}}$
	D_{max}	$0.0277^{\{2\}}$	$0.0287^{\{4\}}$	$0.0291^{\{5\}}$	$0.0273^{\{1\}}$	$0.03^{\{8\}}$	$0.4948^{\{14\}}$	$0.0285^{\{3\}}$	$0.0297^{\{7\}}$	$0.0304^{\{9\}}$	$0.0322^{\{10\}}$	$0.0344^{\{11\}}$	$0.9401^{\{15\}}$	$0.0359^{\{12\}}$	$0.0293^{\{6\}}$	$0.0359^{\{13\}}$
	$\sum$Ranks	$22^{\{1\}}$	$54^{\{4\}}$	$84^{\{8\}}$	$35^{\{2\}}$	$94^{\{10\}}$	$118^{\{11\}}$	$62^{\{5\}}$	$92^{\{9\}}$	$72^{\{6\}}$	$76^{\{7\}}$	$139^{\{13\}}$	$162^{\{15\}}$	$126^{\{14\}}$	$51^{\{3\}}$	$133^{\{12\}}$
350	$\lvert Bias\rvert(\hat{\theta})$	$0.1091^{\{2\}}$	$0.1275^{\{6\}}$	$0.173^{\{9\}}$	$0.1111^{\{4\}}$	$0.175^{\{10\}}$	$0.1304^{\{7\}}$	$0.1479^{\{8\}}$	$0.2013^{\{13\}}$	$0.0853^{\{1\}}$	$0.1104^{\{3\}}$	$0.2151^{\{14\}}$	$1.157^{\{15\}}$	$0.1981^{\{12\}}$	$0.115^{\{5\}}$	$0.1951^{\{11\}}$
	$\lvert Bias\rvert(\hat{\lambda})$	$0.4262^{\{2\}}$	$0.501^{\{6\}}$	$0.6166^{\{10\}}$	$0.4308^{\{3\}}$	$0.6285^{\{11\}}$	$1.702^{\{15\}}$	$0.5178^{\{7\}}$	$0.7316^{\{12\}}$	$0.2985^{\{1\}}$	$0.4668^{\{4\}}$	$1.027^{\{13\}}$	$1.441^{\{14\}}$	$0.5267^{\{8\}}$	$0.4702^{\{5\}}$	$0.5418^{\{9\}}$
	$\lvert Bias\rvert(\hat{\omega})$	$0.0296^{\{4\}}$	$0.0285^{\{1\}}$	$0.0326^{\{9\}}$	$0.0294^{\{2\}}$	$0.0313^{\{7\}}$	$0.5212^{\{14\}}$	$0.0323^{\{8\}}$	$0.0359^{\{10\}}$	$0.0303^{\{5\}}$	$0.0295^{\{3\}}$	$0.0401^{\{11\}}$	$1.634^{\{15\}}$	$0.0445^{\{12\}}$	$0.0304^{\{6\}}$	$0.0446^{\{13\}}$
	MSE ($\hat{\theta}$)	$0.0242^{\{3\}}$	$0.0349^{\{6\}}$	$0.0692^{\{8\}}$	$0.0254^{\{4\}}$	$0.0706^{\{9\}}$	$0.0731^{\{10\}}$	$0.0436^{\{7\}}$	$0.0924^{\{11\}}$	$0.0139^{\{1\}}$	$0.018^{\{2\}}$	$0.1058^{\{14\}}$	$1.341^{\{15\}}$	$0.1001^{\{13\}}$	$0.0262^{\{5\}}$	$0.0941^{\{12\}}$
	MSE ($\hat{\lambda}$)	$0.2468^{\{2\}}$	$0.3409^{\{6\}}$	$0.5742^{\{10\}}$	$0.2568^{\{3\}}$	$0.5817^{\{11\}}$	$2.895^{\{15\}}$	$0.3904^{\{7\}}$	$0.8192^{\{12\}}$	$0.1925^{\{1\}}$	$0.3158^{\{5\}}$	$1.491^{\{13\}}$	$2.089^{\{14\}}$	$0.4357^{\{8\}}$	$0.2858^{\{4\}}$	$0.4536^{\{9\}}$
	MSE ($\hat{\omega}$)	$0.0013^{\{1\}}$	$0.0014^{\{3\}}$	$0.0019^{\{9\}}$	$0.0014^{\{2\}}$	$0.0018^{\{8\}}$	$0.2929^{\{14\}}$	$0.0017^{\{7\}}$	$0.002^{\{10\}}$	$0.0014^{\{4\}}$	$0.0014^{\{5\}}$	$0.0024^{\{11\}}$	$2.672^{\{15\}}$	$0.0029^{\{13\}}$	$0.0015^{\{6\}}$	$0.0029^{\{12\}}$
	MRE ($\hat{\theta}$)	$0.4363^{\{2\}}$	$0.51^{\{6\}}$	$0.6918^{\{9\}}$	$0.4442^{\{4\}}$	$0.6999^{\{10\}}$	$0.5215^{\{7\}}$	$0.5917^{\{8\}}$	$0.8053^{\{13\}}$	$0.3412^{\{1\}}$	$0.4417^{\{3\}}$	$0.8604^{\{14\}}$	$4.63^{\{15\}}$	$0.7923^{\{12\}}$	$0.4599^{\{5\}}$	$0.7806^{\{11\}}$
	MRE ($\hat{\lambda}$)	$0.2436^{\{2\}}$	$0.2863^{\{6\}}$	$0.3523^{\{10\}}$	$0.2462^{\{3\}}$	$0.3592^{\{11\}}$	$0.9723^{\{15\}}$	$0.2959^{\{7\}}$	$0.4181^{\{12\}}$	$0.1705^{\{1\}}$	$0.2667^{\{4\}}$	$0.587^{\{13\}}$	$0.8236^{\{14\}}$	$0.301^{\{8\}}$	$0.2687^{\{5\}}$	$0.3096^{\{9\}}$
	MRE ($\hat{\omega}$)	$0.0592^{\{4\}}$	$0.0569^{\{1\}}$	$0.0652^{\{9\}}$	$0.0588^{\{2\}}$	$0.0626^{\{7\}}$	$1.042^{\{14\}}$	$0.0645^{\{8\}}$	$0.0719^{\{10\}}$	$0.0606^{\{5\}}$	$0.0589^{\{3\}}$	$0.0802^{\{11\}}$	$3.268^{\{15\}}$	$0.089^{\{12\}}$	$0.0607^{\{6\}}$	$0.0893^{\{13\}}$
	D_{abs}	$0.0126^{\{2\}}$	$0.0127^{\{3\}}$	$0.0133^{\{6\}}$	$0.0125^{\{1\}}$	$0.0131^{\{5\}}$	$0.2543^{\{14\}}$	$0.0133^{\{7\}}$	$0.0134^{\{8\}}$	$0.0166^{\{11\}}$	$0.0157^{\{10\}}$	$0.0147^{\{9\}}$	$0.4917^{\{15\}}$	$0.0176^{\{13\}}$	$0.0128^{\{4\}}$	$0.0174^{\{12\}}$
	D_{max}	$0.0227^{\{2\}}$	$0.0229^{\{3\}}$	$0.0242^{\{7\}}$	$0.0223^{\{1\}}$	$0.0241^{\{6\}}$	$0.5062^{\{14\}}$	$0.024^{\{5\}}$	$0.0248^{\{8\}}$	$0.028^{\{10\}}$	$0.0267^{\{9\}}$	$0.0284^{\{11\}}$	$0.9395^{\{15\}}$	$0.0315^{\{13\}}$	$0.0229^{\{4\}}$	$0.0311^{\{12\}}$
	$\sum$Ranks	$26^{\{1\}}$	$47^{\{4\}}$	$96^{\{9\}}$	$29^{\{2\}}$	$95^{\{8\}}$	$139^{\{14\}}$	$79^{\{7\}}$	$119^{\{10\}}$	$41^{\{3\}}$	$51^{\{5\}}$	$134^{\{13\}}$	$162^{\{15\}}$	$124^{\{14\}}$	$55^{\{6\}}$	$123^{\{11\}}$

TABLE VI. Partial and overall ranks for all estimation methods of the ARS-W model from V.

n	Metric	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
	$\lvert Bias\rvert(\hat{\theta})$	6	12	4	7	5	1	15	14	2	3	13	10	8	11	9
	$\lvert Bias\rvert(\hat{\lambda})$	13	6	7	4	8	11	12	14	1	2	15	10	3	9	5
	$\lvert Bias\rvert(\hat{\omega})$	2	5	9	4	8	14	7	10	3	1	13	15	11	6	12
	MSE $(\hat{\theta})$	6	14	4	7	5	1	15	13	2	3	10	11	9	12	8
	MSE $(\hat{\lambda})$	14	8	12	5	9	2	11	13	1	6	15	3	4	7	10
15	MSE $(\hat{\omega})$	3	5	12	4	8	14	7	11	2	1	13	15	9	6	10
	MRE $(\hat{\theta})$	6	12	4	7	5	1	15	14	2	3	13	10	8	11	9
	MRE $(\hat{\lambda})$	13	6	7	4	8	11	12	14	1	2	15	10	3	9	5
	MRE $(\hat{\omega})$	2	5	9	4	8	14	7	10	3	1	13	15	11	6	12
	D_{abs}	1	2	5	3	9	14	4	7	10	6	11	15	12	8	13
	D_{max}	5	4	8	1	7	14	2	9	10	3	13	15	11	6	12
	$\sum$Ranks	$71^{(4)}$	$79^{(5)}$	$81^{(7)}$	$50^{(3)}$	$80^{(6)}$	$97^{(10)}$	$107^{(12)}$	$129^{(13.5)}$	$37^{(2)}$	$31^{(1)}$	$144^{(15)}$	$129^{(13.5)}$	$89^{(8)}$	$91^{(9)}$	$105^{(11)}$
	$\lvert Bias\rvert(\hat{\theta})$	4	7	9	12	10	1	5	13	2	3	14	8	6	11	15
	$\lvert Bias\rvert(\hat{\lambda})$	2	6	11	5	9	14	8	12	1	3	15	13	7	4	10
	$\lvert Bias\rvert(\hat{\omega})$	1	6	11	4	12	14	7	8	3	5	13	15	9	2	10
	MSE $(\hat{\theta})$	4	8	9	11	10	1	6	13	2	3	14	5	7	12	15
	MSE $(\hat{\lambda})$	4	6	12	5	10	13	8	14	1	2	15	11	7	3	9
50	MSE $(\hat{\omega})$	1	6	10	5	11	14	7	8	2	4	13	15	9	3	12
	MRE $(\hat{\theta})$	4	7	9	12	10	1	5	13	2	3	14	8	6	11	15
	MRE $(\hat{\lambda})$	2	6	11	5	9	14	8	12	1	3	15	13	7	4	10
	MRE $(\hat{\omega})$	1	6	11	4	12	14	7	8	3	5	13	15	9	2	10
	D_{abs}	1	4	8	2	7	14	3	6	11	9	10	15	12	5	13
	D_{max}	1	4	8	2	6	14	3	7	10	9	11	15	12	5	13
	$\sum$Ranks	$25^{(1)}$	$66^{(5)}$	$109^{(10)}$	$67^{(6.5)}$	$106^{(9)}$	$114^{(11.5)}$	$67^{(6.5)}$	$114^{(11.5)}$	$38^{(2)}$	$49^{(3)}$	$147^{(15)}$	$133^{(14)}$	$91^{(8)}$	$62^{(4)}$	$132^{(13)}$
	$\lvert Bias\rvert(\hat{\theta})$	3	8	9	12	11	1	6	13	2	4	7	10	15	5	14
	$\lvert Bias\rvert(\hat{\lambda})$	2	6	11	5	8	15	7	12	1	4	13	14	9	3	10
	$\lvert Bias\rvert(\hat{\omega})$	1	5	9	7	11	14	6	8	2	3	10	15	12	4	13
	MSE $(\hat{\theta})$	3	10	9	12	11	1	6	15	2	5	7	8	14	4	13
	MSE $(\hat{\lambda})$	2	6	11	5	10	14	7	12	1	4	15	13	8	3	9
120	MSE $(\hat{\omega})$	2	6	9	7	11	14	5	10	1	3	8	15	12	4	13
	MRE $(\hat{\theta})$	3	8	9	12	11	1	6	13	2	4	7	10	15	5	14
	MRE $(\hat{\lambda})$	2	6	11	5	8	15	7	12	1	4	13	14	9	3	10
	MRE $(\hat{\omega})$	1	5	9	7	11	14	6	8	2	3	10	15	12	4	13
	D_{abs}	1	3	7	4	6	14	2	5	10	8	9	15	13	11	12
	D_{max}	1	4	9	2	7	14	3	5	8	6	11	15	13	10	12
	$\sum$Ranks	$21^{(1)}$	$67^{(6)}$	$103^{(8)}$	$78^{(7)}$	$105^{(9)}$	$117^{(12)}$	$61^{(5)}$	$113^{(11)}$	$32^{(2)}$	$48^{(3)}$	$110^{(10)}$	$144^{(15)}$	$132^{(13)}$	$56^{(4)}$	$133^{(14)}$
	$\lvert Bias\rvert(\hat{\theta})$	4	8	11	6	10	1	7	9	2	5	14	15	12	3	13
	$\lvert Bias\rvert(\hat{\lambda})$	2	6	9	4	8	15	7	12	1	5	14	13	11	3	10
	$\lvert Bias\rvert(\hat{\omega})$	1	6	11	2	9	14	5	7	4	8	10	15	12	3	13
	MSE $(\hat{\theta})$	5	8	10	6	11	1	7	9	2	4	15	14	12	3	13
	MSE $(\hat{\lambda})$	3	6	9	4	8	14	7	12	1	5	15	13	11	2	10
180	MSE $(\hat{\omega})$	1	7	10	2	9	14	5	6	4	8	11	15	12	3	13
	MRE $(\hat{\theta})$	4	8	11	6	10	1	7	9	2	5	14	15	12	3	13
	MRE $(\hat{\lambda})$	2	6	9	4	8	15	7	12	1	5	14	13	11	3	10
	MRE $(\hat{\omega})$	1	6	11	2	9	14	5	7	4	8	10	15	12	3	13
	D_{abs}	2	3	5	1	6	14	4	7	10	8	12	15	13	9	11
	D_{max}	2	3	6	1	5	14	4	7	9	8	13	15	12	10	11
	$\sum$Ranks	$27^{(1)}$	$67^{(6)}$	$102^{(10)}$	$38^{(2)}$	$93^{(8)}$	$117^{(11)}$	$65^{(5)}$	$97^{(9)}$	$40^{(3)}$	$69^{(7)}$	$142^{(14)}$	$158^{(15)}$	$130^{(12.5)}$	$45^{(4)}$	$130^{(12.5)}$
	$\lvert Bias\rvert(\hat{\theta})$	2	7	9	4	10	1	8	11	6	5	14	15	12	3	13
	$\lvert Bias\rvert(\hat{\lambda})$	3	6	9	4	8	15	7	12	1	5	13	14	11	2	10
	$\lvert Bias\rvert(\hat{\omega})$	1	2	6	5	8	14	3	4	10	9	13	15	11	7	12
	MSE $(\hat{\theta})$	2	7	10	3	11	1	8	9	6	4	12	15	13	5	14
	MSE $(\hat{\lambda})$	3	6	9	2	8	15	7	12	5	4	13	14	10	1	11
250	MSE $(\hat{\omega})$	1	3	7	2	8	14	5	4	10	9	13	15	11	6	12
	MRE $(\hat{\theta})$	2	7	9	4	10	1	8	11	6	5	14	15	12	3	13
	MRE $(\hat{\lambda})$	3	6	9	4	8	15	7	12	1	5	13	14	11	2	10
	MRE $(\hat{\omega})$	1	2	6	5	8	14	3	4	10	9	13	15	11	7	12
	D_{abs}	2	4	5	1	7	14	3	6	8	11	10	15	12	9	13
	D_{max}	2	4	5	1	8	14	3	7	9	10	11	15	12	6	13
	$\sum$Ranks	$22^{(1)}$	$54^{(4)}$	$84^{(8)}$	$35^{(2)}$	$94^{(10)}$	$118^{(11)}$	$62^{(5)}$	$92^{(9)}$	$72^{(6)}$	$76^{(7)}$	$139^{(14)}$	$162^{(15)}$	$126^{(12)}$	$51^{(3)}$	$133^{(13)}$
	$\lvert Bias\rvert(\hat{\theta})$	2	6	9	4	10	7	8	13	1	3	14	15	12	5	11
	$\lvert Bias\rvert(\hat{\lambda})$	2	6	10	3	11	15	7	12	1	4	13	14	8	5	9
	$\lvert Bias\rvert(\hat{\omega})$	4	1	9	2	7	14	8	10	5	3	11	15	12	6	13
	MSE $(\hat{\theta})$	3	6	8	4	9	10	7	11	1	2	14	15	13	5	12
	MSE $(\hat{\lambda})$	2	6	10	3	11	15	7	12	1	5	13	14	8	4	9
350	MSE $(\hat{\omega})$	1	3	9	2	8	14	7	10	4	5	11	15	13	6	12
	MRE $(\hat{\theta})$	2	6	9	4	10	7	8	13	1	3	14	15	12	5	11
	MRE $(\hat{\lambda})$	2	6	10	3	11	15	7	12	1	4	13	14	8	5	9
	MRE $(\hat{\omega})$	4	1	9	2	7	14	8	10	5	3	11	15	12	6	13
	D_{abs}	2	3	6	1	5	14	7	8	11	10	9	15	13	4	12
	D_{max}	2	3	7	1	6	14	5	8	10	9	11	15	13	4	12
	$\sum$Ranks	$26^{(1)}$	$47^{(4)}$	$96^{(9)}$	$29^{(2)}$	$95^{(8)}$	$139^{(14)}$	$79^{(7)}$	$119^{(10)}$	$41^{(3)}$	$51^{(5)}$	$134^{(13)}$	$162^{(15)}$	$124^{(12)}$	$55^{(6)}$	$123^{(11)}$

Table V summarizes a simulation study comparing the performance of sixteen different estimation methods for the parameters ($\theta = 0.25, \lambda = 1.75, \omega = 0.5$) of a distribution, using five common sample sizes ($n = 15, 50, 120, 180, 250, 350$). The performance is evaluated based on three accuracy metrics: absolute bias ($|Bias|$), MSE, and MRE, as well as two goodness-of-fit statistics, D_{abs} and D_{max}. Each cell contains the metric value followed by its rank among the sixteen estimators, with a lower rank indicating better performance for that specific metric and parameter combination. Overall, the Log-Moment Estimators (MSALDE and MSALDE) generally show the best performance. Specifically, MSALDE (Mean Squared Absolute Logarithmic Distance Estimator) is the top-ranked estimator based on the cumulative rank ($\sum$Ranks) for all sample sizes, consistently outperforming the other methods. MSALDE also often yields the lowest absolute bias and mean squared error for $\hat{\lambda}$ and $\hat{\omega}$. The MLE and the MSALDE are frequently among the top five best estimators, particularly for smaller sample sizes, although the MLE is notably less effective at estimating θ for $n = 15$. As the sample size (n) increases from 15 to 350, the performance of all estimators improves, which is evident as the values for $|Bias|$, MSE, and MRE consistently decrease. The increase in sample size also tends to stabilize the performance ranking across the different metrics, confirming the theoretical property of consistency for most of these estimators. The RTADE and the KE consistently rank among the worst performers across nearly all metrics and sample sizes, often having the highest values for bias, MSE, and MRE.

Table VI presents the partial and overall ranks for sixteen different estimation methods used in a simulation study of the ARS-W model contained in Table V, for sample sizes (n) ranging from 15 to 350. The ranking is based on various performance metrics, including absolute bias ($|Bias|$), MSE, MRE for the three parameters (θ, λ, ω), and two overall goodness-of-fit statistics (D_{abs}, D_{max}). For all metrics, a rank of 1 indicates the best performance, and 15 is the worst. The MSALDE consistently demonstrates the best overall performance, achieving the lowest cumulative rank ($\sum$Ranks) across all sample sizes. This indicates it is the most reliable method overall for estimating the model parameters. The MPSE and the MSADE are also highly competitive, often securing the second or third best overall rank. The MLE is a strong performer, consistently ranking in the top five overall, with its rank generally improving as the sample size increases. Conversely, the KE and the ADSOE are almost always ranked among the worst methods, indicating poor performance regardless of the sample size. As the sample size (n) increases, the cumulative ranks for the best-performing methods generally decrease, indicating greater convergence and improved relative performance. This trend confirms the expected improvement in estimation accuracy as more data becomes available. The RTADE performs exceptionally well in estimating the θ parameter, frequently achieving rank 1, but its overall rank is dragged down by poor performance in other areas. Figures 9, 11, 10 and 12 are plots for the Bias, MSE, MRE, D_{abs} and D_{max} which support the results in Table V.

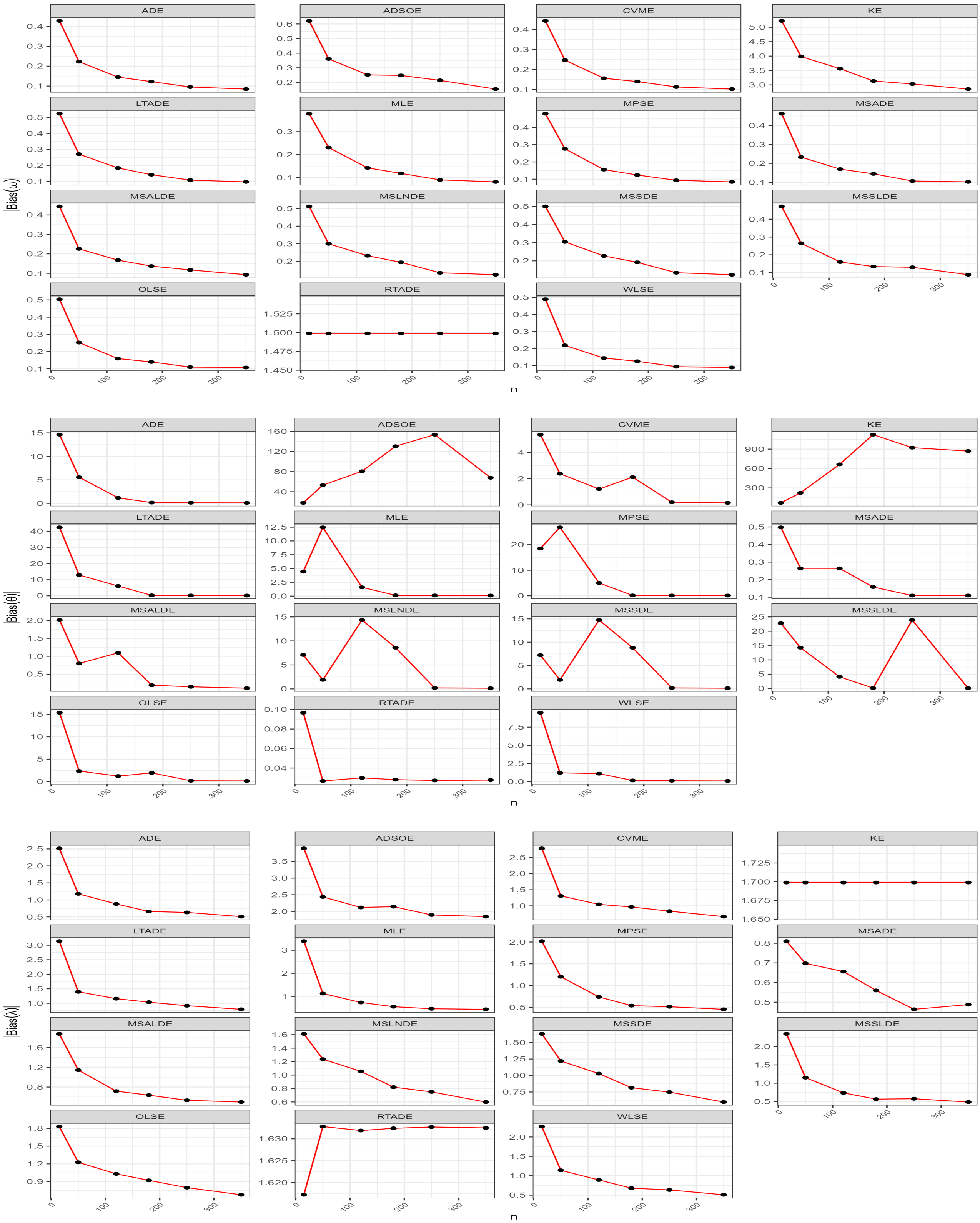

Fig. 13. Graphical representations for the Bias values presented in Table VII.

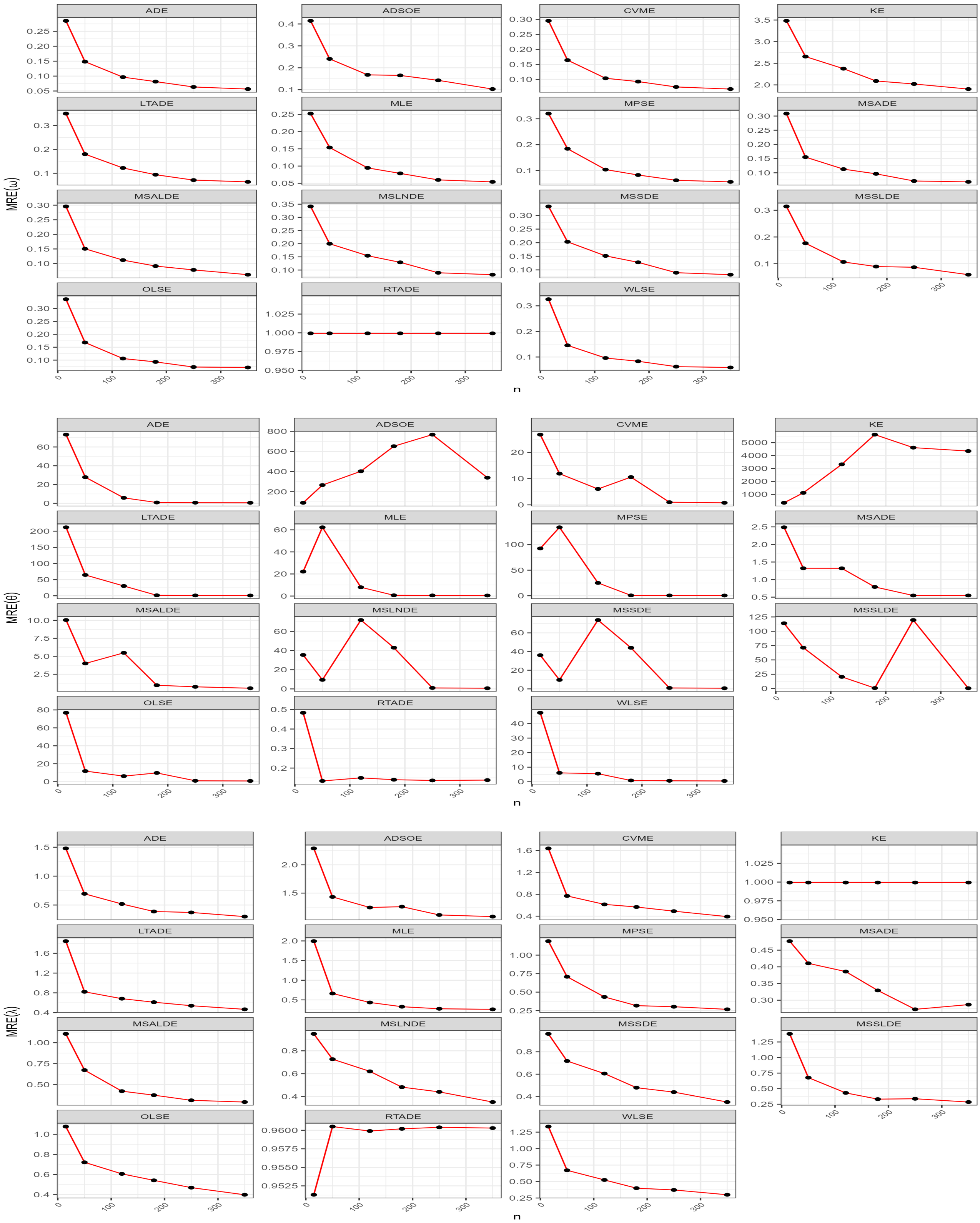

Fig. 14. Graphical representations for the MRE values presented in Table VII.

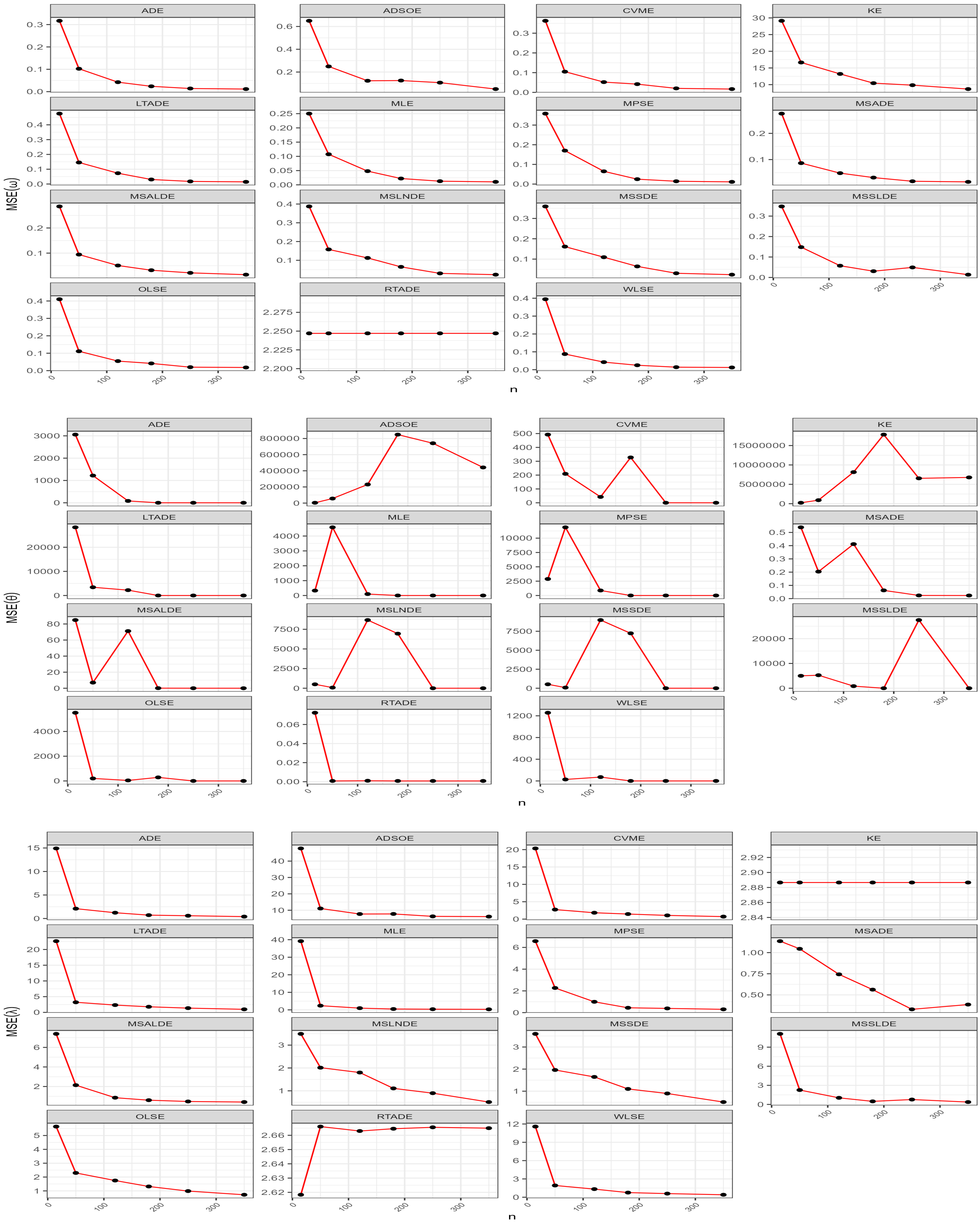

Fig. 15. Graphical representations for the MSE values presented in Table VII.

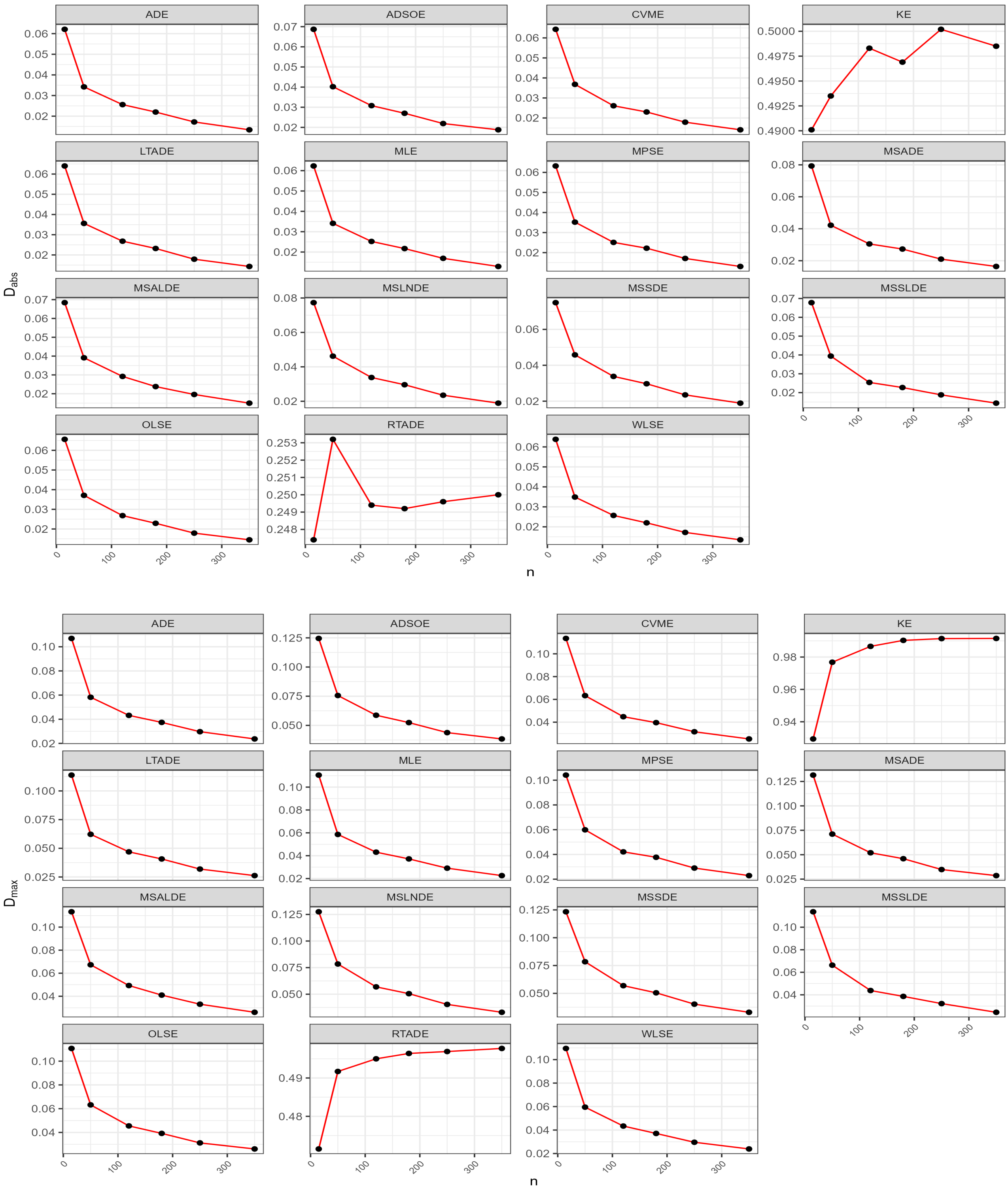

Fig. 16. Graphical representations for the D_{abs} and D_{max} values presented in Table VII.

TABLE VII. Numerical values of simulation measures for $\theta = 0.25,\ \lambda = 1.75\ \omega = 0.5$.

n	Metric	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
15	$\|Bias\|(\hat{\theta})$	4.4305[4]	14.6348[9]	5.3543[5]	18.4933[12]	15.3652[10]	0.0967[1]	9.4816[8]	42.3937[14]	0.4975[2]	2.0096[3]	17.9082[11]	70.257[15]	7.2165[7]	22.7877[13]	7.0792[6]
	$\|Bias\|(\hat{\lambda})$	3.3904[14]	2.5152[11]	2.785[12]	2.0211[8]	1.8309[6]	1.6172[3]	2.2682[9]	3.1391[13]	0.8109[1]	1.8784[7]	3.8925[15]	1.699[5]	1.6317[4]	2.3466[10]	1.612[2]
	$\|Bias\|(\hat{\omega})$	0.3785[1]	0.4278[2]	0.4427[3]	0.4799[7]	0.5039[10]	1.499[14]	0.4888[8]	0.5246[12]	0.4627[5]	0.4436[4]	0.6215[13]	5.2226[15]	0.4999[9]	0.4715[6]	0.5123[11]
	MSE $(\hat{\theta})$	330.0645[4]	3055.2125[10]	492.3922[5]	2883.8567[9]	5511.3136[13]	0.0723[1]	1256.0564[8]	28259.3791[14]	0.5385[2]	84.8884[3]	3762.5182[11]	236546.1736[15]	517.7025[7]	4998.2636[12]	498.0353[6]
	MSE $(\hat{\lambda})$	39.1617[14]	14.9043[11]	20.3401[12]	6.5769[7]	5.6446[6]	2.6183[2]	11.5751[10]	22.6526[13]	1.1357[1]	7.3854[8]	47.7313[15]	2.8866[3]	3.5951[5]	11.0893[9]	3.4923[4]
	MSE $(\hat{\omega})$	0.2501[1]	0.3168[4]	0.3627[8]	0.3582[6]	0.41[11]	2.247[14]	0.3944[10]	0.4757[12]	0.2753[2]	0.2857[3]	0.6497[13]	29.1269[15]	0.3591[7]	0.3472[5]	0.3872[9]
	MRE $(\hat{\theta})$	22.1525[4]	73.1738[9]	26.7714[5]	92.4667[12]	76.826[10]	0.4833[1]	47.4079[8]	211.9683[14]	2.4877[2]	10.0478[3]	89.5409[11]	351.2849[15]	36.0823[7]	113.9383[13]	35.3962[6]
	MRE $(\hat{\lambda})$	1.9943[14]	1.4795[11]	1.6382[12]	1.1889[8]	1.077[6]	0.9513[3]	1.3342[9]	1.8465[13]	0.477[1]	1.1049[7]	2.2897[15]	0.9994[5]	0.9598[4]	1.3803[10]	0.9482[2]
	MRE $(\hat{\omega})$	0.2523[1]	0.2852[2]	0.2952[3]	0.3199[7]	0.3359[10]	0.9993[14]	0.3258[8]	0.3497[12]	0.3085[5]	0.2957[4]	0.4144[13]	3.4817[15]	0.3333[9]	0.3143[6]	0.3415[11]
	D_{abs}	0.0623[2]	0.0621[1]	0.0643[6]	0.0632[3]	0.0656[7]	0.2474[14]	0.0638[4]	0.064[5]	0.0793[13]	0.0684[9]	0.0687[10]	0.4901[15]	0.0749[11]	0.0678[8]	0.0773[12]
	D_{max}	0.1105[4]	0.1069[2]	0.1135[7]	0.1041[1]	0.1107[5]	0.4715[14]	0.1095[3]	0.1138[9]	0.1315[13]	0.1133[6]	0.1245[11]	0.9294[15]	0.1233[10]	0.1137[8]	0.1275[12]
	$\sum$Ranks	63[3]	72[4]	78[5]	80[6]	94[11]	81[8]	85[10]	131[13]	47[1]	57[2]	138[15]	133[14]	80[6]	100[12]	81[8]
50	$\|Bias\|(\hat{\theta})$	12.4557[10]	5.5657[9]	2.377[8]	26.7566[13]	2.3667[7]	0.0268[1]	1.2196[4]	12.8887[11]	0.2644[2]	0.7998[3]	53.1055[14]	223.6838[15]	1.9256[6]	14.2719[12]	1.8892[5]
	$\|Bias\|(\hat{\lambda})$	1.1258[2]	1.1799[6]	1.3091[11]	1.2035[7]	1.2266[9]	1.6328[13]	1.1396[3]	1.396[12]	0.6978[1]	1.1441[4]	2.4352[15]	1.699[14]	1.2201[8]	1.1517[5]	1.2359[10]
	$\|Bias\|(\hat{\omega})$	0.2309[4]	0.2225[2]	0.2463[6]	0.2762[10]	0.2524[7]	1.499[14]	0.2184[1]	0.2702[9]	0.2329[5]	0.2258[3]	0.361[13]	3.9825[15]	0.3047[12]	0.2649[8]	0.2996[11]
	MSE $(\hat{\theta})$	4585.3137[11]	1219.518[9]	209.0967[8]	11886.6368[13]	202.741[7]	0.0008[1]	27.5708[4]	3415.6644[10]	0.2039[2]	7.0208[3]	56325.7111[14]	910246.8327[15]	84.9406[6]	5251.1115[12]	81.4681[5]
	MSE $(\hat{\lambda})$	2.3874[10]	2.0921[5]	2.7444[12]	2.2692[8]	2.2975[9]	2.6661[11]	1.9264[2]	3.2144[14]	1.0448[1]	2.137[6]	11.063[15]	2.8866[13]	1.9603[3]	2.2625[7]	2.0128[4]
	MSE $(\hat{\omega})$	0.1074[6]	0.1024[4]	0.1049[5]	0.1701[12]	0.1113[7]	2.247[14]	0.0876[2]	0.1456[8]	0.0865[1]	0.0943[3]	0.2493[13]	16.6442[15]	0.1615[11]	0.1482[9]	0.158[10]
	MRE $(\hat{\theta})$	62.2783[10]	27.8285[9]	11.8852[8]	133.783[13]	11.8334[7]	0.134[1]	6.0979[4]	64.4435[11]	1.3219[2]	3.9989[3]	265.5273[14]	1118.4192[15]	9.6279[6]	71.3596[12]	9.446[5]
	MRE $(\hat{\lambda})$	0.6622[2]	0.6941[6]	0.7701[11]	0.7079[7]	0.7215[9]	0.9605[13]	0.6704[3]	0.8212[12]	0.4105[1]	0.673[4]	1.4325[15]	0.9994[14]	0.7177[8]	0.6774[5]	0.727[10]
	MRE $(\hat{\omega})$	0.1539[4]	0.1483[2]	0.1642[6]	0.1841[10]	0.1683[7]	0.9993[14]	0.1456[1]	0.1801[9]	0.1553[5]	0.1506[3]	0.2406[13]	2.655[15]	0.2032[12]	0.1766[8]	0.1998[11]
	D_{abs}	0.0341[1]	0.0342[2]	0.0368[6]	0.0352[4]	0.0371[7]	0.2532[14]	0.0349[3]	0.0356[5]	0.0422[11]	0.0391[8]	0.0402[10]	0.4935[15]	0.0458[12]	0.0394[9]	0.0462[13]
	D_{max}	0.0586[2]	0.0582[1]	0.0633[7]	0.0599[4]	0.0632[6]	0.4917[14]	0.0595[3]	0.0622[5]	0.0711[10]	0.0673[9]	0.0756[11]	0.9768[15]	0.0784[12]	0.0663[8]	0.0784[13]
	$\sum$Ranks	62[5]	55[4]	88[7]	101[11]	82[6]	110[13]	30[1]	106[12]	41[2]	49[3]	147[14]	161[15]	96[9]	95[8]	97[10]
120	$\|Bias\|(\hat{\theta})$	1.5872[8]	1.1625[5]	1.2101[6]	5.0252[10]	1.2364[7]	0.0299[1]	1.1089[4]	6.0656[11]	0.2641[2]	1.0963[3]	80.5969[14]	664.3603[15]	14.7482[13]	4.0812[9]	14.3582[12]
	$\|Bias\|(\hat{\lambda})$	0.7394[5]	0.8813[6]	1.0471[10]	0.7392[4]	1.0313[9]	1.6319[13]	0.8913[7]	1.1588[12]	0.6558[1]	0.7172[2]	2.1175[15]	1.699[14]	1.0289[8]	0.7348[3]	1.0549[11]
	$\|Bias\|(\hat{\omega})$	0.142[1]	0.1448[3]	0.1557[5]	0.1555[4]	0.1592[6]	1.499[14]	0.1441[2]	0.1832[10]	0.1693[9]	0.1677[8]	0.2517[13]	3.5621[15]	0.2273[11]	0.1599[7]	0.2317[12]
	MSE $(\hat{\theta})$	97.8313[8]	83.0873[7]	42.4728[4]	874.3107[10]	42.3397[3]	0.001[1]	71.1307[5]	2221.2319[11]	0.4117[2]	71.1689[6]	230740.7978[14]	8145810.074[15]	8976.6746[13]	845.4095[9]	8676.6764[12]
	MSE $(\hat{\lambda})$	0.9863[3]	1.2298[6]	1.8318[11]	0.9954[4]	1.7484[9]	2.663[13]	1.331[7]	2.3254[12]	0.7433[1]	0.8533[2]	7.6828[15]	2.8866[14]	1.6455[8]	1.0363[5]	1.8003[10]
	MSE $(\hat{\omega})$	0.0481[3]	0.0424[2]	0.0521[6]	0.0648[9]	0.0548[7]	2.247[14]	0.0424[1]	0.073[10]	0.0482[4]	0.0507[5]	0.1232[13]	13.2162[15]	0.1094[11]	0.0573[8]	0.1126[12]
	MRE $(\hat{\theta})$	7.9358[8]	5.8126[5]	6.0506[6]	25.1258[10]	6.1818[7]	0.1497[1]	5.5445[4]	30.3281[11]	1.3203[2]	5.4816[3]	402.9845[14]	3321.8014[15]	73.7411[13]	20.4061[9]	71.791[12]
	MRE $(\hat{\lambda})$	0.4349[5]	0.5184[6]	0.616[10]	0.4348[4]	0.6066[9]	0.9599[13]	0.5243[7]	0.6817[12]	0.3858[1]	0.4219[2]	1.2456[15]	0.9994[14]	0.6053[8]	0.4323[3]	0.6205[11]
	MRE $(\hat{\omega})$	0.0947[1]	0.0965[3]	0.1038[5]	0.1037[4]	0.1061[6]	0.9993[14]	0.0961[2]	0.1222[10]	0.1129[9]	0.1118[8]	0.1678[13]	2.3747[15]	0.1515[11]	0.1066[7]	0.1544[12]
	D_{abs}	0.0252[2]	0.0256[4]	0.0261[6]	0.0251[1]	0.0268[8]	0.2494[14]	0.0257[5]	0.0268[7]	0.0305[10]	0.0292[9]	0.0308[11]	0.4983[15]	0.0338[13]	0.0254[3]	0.0338[12]
	D_{max}	0.0431[2]	0.0432[3]	0.0448[6]	0.0421[1]	0.0455[7]	0.495[14]	0.0434[4]	0.0469[8]	0.052[10]	0.0493[9]	0.0587[13]	0.9866[15]	0.0569[11]	0.0438[5]	0.0569[12]
	$\sum$Ranks	46[1]	50[3]	75[8]	61[6]	78[9]	112[10]	48[2]	114[11]	51[4]	57[5]	150[14]	162[15]	120[12]	68[7]	128[13]
180	$\|Bias\|(\hat{\theta})$	0.1333[2]	0.1695[6]	2.1153[11]	0.1402[3]	1.9475[10]	0.0281[1]	0.1725[7]	0.2941[9]	0.1581[4]	0.1933[8]	130.352[14]	1123.8302[15]	8.8073[13]	0.1627[5]	8.5966[12]
	$\|Bias\|(\hat{\lambda})$	0.5558[2]	0.6591[6]	0.9655[11]	0.5398[1]	0.9214[10]	1.6324[13]	0.6793[7]	1.038[12]	0.5598[3]	0.636[5]	2.1418[15]	1.699[14]	0.8152[8]	0.5644[4]	0.8202[9]
	$\|Bias\|(\hat{\omega})$	0.1181[1]	0.1222[2]	0.1391[7]	0.1241[3]	0.1397[8]	1.499[14]	0.1251[4]	0.141[9]	0.1444[10]	0.1371[6]	0.2476[13]	3.1351[15]	0.1918[11]	0.1341[5]	0.1938[12]
	MSE $(\hat{\theta})$	0.0454[2]	0.092[5]	327.4605[11]	0.0529[3]	287.4258[10]	0.0008[1]	0.0995[6]	0.2772[9]	0.0622[4]	0.1191[8]	848225.8081[14]	17822807.9791[15]	7227.4691[13]	0.1012[7]	6942.9364[12]
	MSE $(\hat{\lambda})$	0.4861[3]	0.7162[6]	1.4617[11]	0.4441[1]	1.3188[10]	2.6646[13]	0.7592[7]	1.7809[12]	0.5622[4]	0.5994[5]	7.7392[15]	2.8866[14]	1.1019[8]	0.4811[2]	1.1073[9]
	MSE $(\hat{\omega})$	0.0221[1]	0.0239[2]	0.0418[10]	0.0249[3]	0.0414[9]	2.247[14]	0.0249[4]	0.0305[5]	0.0312[7]	0.032[8]	0.1253[13]	10.449[15]	0.0641[11]	0.0306[6]	0.0649[12]
	MRE $(\hat{\theta})$	0.6667[2]	0.8477[6]	10.5765[11]	0.7011[3]	9.7374[10]	0.1403[1]	0.8627[7]	1.4706[9]	0.7903[4]	0.9663[8]	651.7598[14]	5619.1509[15]	44.0363[13]	0.8133[5]	42.9831[12]
	MRE $(\hat{\lambda})$	0.3269[2]	0.3877[6]	0.5679[11]	0.3175[1]	0.542[10]	0.9602[13]	0.3996[7]	0.6106[12]	0.3293[3]	0.3741[5]	1.2599[15]	0.9994[14]	0.4796[8]	0.332[4]	0.4825[9]
	MRE $(\hat{\omega})$	0.0787[1]	0.0815[2]	0.0927[7]	0.0827[3]	0.0931[8]	0.9993[14]	0.0834[4]	0.094[9]	0.0963[10]	0.0914[6]	0.1651[13]	2.0901[15]	0.1278[11]	0.0894[5]	0.1292[12]
	D_{abs}	0.0217[1]	0.022[3]	0.023[7]	0.0222[4]	0.0229[6]	0.2492[14]	0.022[2]	0.0232[8]	0.0273[11]	0.0238[9]	0.027[10]	0.4969[15]	0.0297[13]	0.0227[5]	0.0296[12]
	D_{max}	0.0372[2]	0.0374[3]	0.0396[7]	0.0377[4]	0.0392[6]	0.4964[14]	0.0371[1]	0.0407[8]	0.0459[10]	0.041[9]	0.0524[13]	0.9903[15]	0.0505[12]	0.0386[5]	0.0505[11]
	$\sum$Ranks	19[1]	47[3]	104[10]	29[2]	97[8]	112[11]	56[5]	102[9]	70[6]	77[7]	149[14]	162[15]	121[12]	53[4]	122[13]
250	$\|Bias\|(\hat{\theta})$	0.1102[3]	0.1357[5]	0.2015[8]	0.1161[4]	0.2021[9]	0.0273[1]	0.1362[6]	0.2115[12]	0.1093[2]	0.1496[7]	153.4373[14]	922.5934[15]	0.2033[10]	23.92[13]	0.2045[11]
	$\|Bias\|(\hat{\lambda})$	0.4674[2]	0.6321[6]	0.836[11]	0.5143[3]	0.7984[10]	1.6327[13]	0.6336[7]	0.9167[12]	0.4637[1]	0.5318[4]	1.893[15]	1.699[14]	0.7485[8]	0.5748[5]	0.7497[9]
	$\|Bias\|(\hat{\omega})$	0.0898[1]	0.0952[4]	0.1122[8]	0.0935[2]	0.1097[7]	1.499[14]	0.0936[3]	0.107[6]	0.1067[5]	0.1174[9]	0.214[13]	3.032[15]	0.1346[12]	0.1301[10]	0.1343[11]
	MSE $(\hat{\theta})$	0.0247[3]	0.0331[5]	0.0947[8]	0.028[4]	0.0986[9]	0.0008[1]	0.0346[6]	0.1155[10]	0.0239[2]	0.0514[7]	741181.2687[14]	6547776.3442[15]	0.1429[11]	27499.6307[13]	0.1461[12]
	MSE $(\hat{\lambda})$	0.3865[2]	0.5904[5]	1.0688[11]	0.3875[3]	0.9872[10]	2.6656[13]	0.5929[6]	1.3562[12]	0.3294[1]	0.4674[4]	6.2585[15]	2.8866[14]	0.8948[8]	0.7667[7]	0.8974[9]
	MSE $(\hat{\omega})$	0.0128[1]	0.0141[3]	0.0203[8]	0.0143[4]	0.0199[7]	2.247[14]	0.0141[2]	0.0176[6]	0.0172[5]	0.0215[9]	0.1069[13]	9.8452[15]	0.0303[11]	0.0491[12]	0.0302[10]
	MRE $(\hat{\theta})$	0.551[3]	0.6784[5]	1.0074[8]	0.5807[4]	1.0104[9]	0.1365[1]	0.6809[6]	1.0573[12]	0.5467[2]	0.7481[7]	767.1865[14]	4612.9669[15]	1.0163[10]	119.6001[13]	1.0225[11]
	MRE $(\hat{\lambda})$	0.2749[2]	0.3718[6]	0.4918[11]	0.3026[3]	0.4696[10]	0.9604[13]	0.3727[7]	0.5393[12]	0.2727[1]	0.3128[4]	1.1135[15]	0.9994[14]	0.4403[8]	0.3381[5]	0.441[9]
	MRE $(\hat{\omega})$	0.0598[1]	0.0635[4]	0.0748[8]	0.0623[2]	0.0732[7]	0.9993[14]	0.0624[3]	0.0714[6]	0.0711[5]	0.0782[9]	0.1427[13]	2.0213[15]	0.0897[12]	0.0867[10]	0.0896[11]
	D_{abs}	0.0169[1]	0.0172[4]	0.0179[5]	0.0171[2]	0.0179[7]	0.2496[14]	0.0172[3]	0.0179[6]	0.021[10]	0.0196[9]	0.0219[11]	0.5002[15]	0.0235[12]	0.0188[8]	0.0235[13]
	D_{max}	0.0291[2]	0.0297[4]	0.0316[6]	0.029[1]	0.0312[5]	0.4969[14]	0.0296[3]	0.0319[7]	0.0348[10]	0.0331[9]	0.0438[13]	0.9914[15]	0.0403[11]	0.0322[8]	0.0404[12]
	$\sum$Ranks	21[1]	51[4]	92[8]	32[2]	90[7]	112[11]	52[5]	101[9]	44[3]	78[6]	150[14]	162[15]	113[12]	104[10]	118[13]
350	$\|Bias\|(\hat{\theta})$	0.0955[2]	0.1112[8]	0.1583[11]	0.0976[3]	0.1626[12]	0.0277[1]	0.11[5]	0.1701[13]	0.1095[4]	0.1106[7]	67.7929[14]	869.5745[15]	0.1441[10]	0.1105[6]	0.1441[9]
	$\|Bias\|(\hat{\lambda})$	0.4429[1]	0.5093[7]	0.6672[10]	0.4549[2]	0.6776[11]	1.6325[13]	0.509[6]	0.7925[12]	0.4876[4]	0.4966[5]	1.8435[15]	1.699[14]	0.5978[8]	0.4851[3]	0.5988[9]
	$\|Bias\|(\hat{\omega})$	0.0811[1]	0.0847[3]	0.1016[8]	0.0843[2]	0.1073[10]	1.499[14]	0.0887[5]	0.0959[7]	0.102[9]	0.0929[6]	0.1543[13]	2.8555[15]	0.1237[12]	0.0885[4]	0.1235[11]
	MSE $(\hat{\theta})$	0.0164[2]	0.0228[4]	0.0561[11]	0.0164[3]	0.058[13]	0.0008[1]	0.0235[6]	0.0579[12]	0.0234[5]	0.0248[7]	441525.9867[14]	6776437.5889[15]	0.0456[9]	0.0268[8]	0.0457[10]
	MSE $(\hat{\lambda})$	0.3099[2]	0.4039[6]	0.7222[11]	0.303[1]	0.7204[10]	2.665[13]	0.4036[5]	0.9709[12]	0.3858[4]	0.4095[7]	6.1065[15]	2.8866[14]	0.5082[8]	0.3734[3]	0.5092[9]
	MSE $(\hat{\omega})$	0.0104[1]	0.0116[3]	0.0168[9]	0.0112[2]	0.0181[10]	2.247[14]	0.0124[4]	0.0145[7]	0.0149[8]	0.0143[6]	0.0505[13]	8.6778[15]	0.0234[12]	0.0136[5]	0.0233[11]
	MRE $(\hat{\theta})$	0.4775[2]	0.5561[8]	0.7917[11]	0.4879[3]	0.8128[12]	0.1383[1]	0.5499[5]	0.8504[13]	0.5475[4]	0.5532[7]	338.9646[14]	4347.8724[15]	0.7207[10]	0.5523[6]	0.7206[9]
	MRE $(\hat{\lambda})$	0.2605[1]	0.2996[7]	0.3924[10]	0.2676[2]	0.3986[11]	0.9603[13]	0.2994[6]	0.4662[12]	0.2868[4]	0.2921[5]	1.0844[15]	0.9994[14]	0.3516[8]	0.2853[3]	0.3522[9]
	MRE $(\hat{\omega})$	0.0541[1]	0.0565[3]	0.0678[8]	0.0562[2]	0.0715[10]	0.9993[14]	0.0591[5]	0.064[7]	0.068[9]	0.0619[6]	0.1029[13]	1.9037[15]	0.0824[12]	0.059[4]	0.0824[11]
	D_{abs}	0.0129[1]	0.0134[3]	0.0141[5]	0.0131[2]	0.0145[8]	0.25[14]	0.0135[4]	0.0143[6]	0.0164[10]	0.015[9]	0.0188[11]	0.4985[15]	0.0189[12]	0.0144[7]	0.0189[13]
	D_{max}	0.0226[1]	0.0237[3]	0.0253[6]	0.0229[2]	0.0261[8]	0.4977[14]	0.0239[4]	0.0262[9]	0.0286[10]	0.0261[7]	0.0384[13]	0.9915[15]	0.033[12]	0.0245[5]	0.033[11]
	$\sum$Ranks	15[1]	55[4]	100[8]	24[2]	115[13]	112[10]	55[4]	110[9]	71[6]	72[7]	150[14]	162[15]	113[12]	54[3]	112[10]

TABLE VIII. Partial and overall ranks for all estimation methods of the ARS-W model from VII.

n	Metric	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
15	$\|Bias\|(\hat{\theta})$	4.0	9.0	5.0	12.0	10.0	1.0	8.0	14.0	2.0	3.0	11.0	15.0	7.0	13.0	6.0
	$\|Bias\|(\hat{\lambda})$	14.0	11.0	12.0	8.0	6.0	3.0	9.0	13.0	1.0	7.0	15.0	5.0	4.0	10.0	2.0
	$\|Bias\|(\hat{\omega})$	1.0	2.0	3.0	7.0	10.0	14.0	8.0	12.0	5.0	4.0	13.0	15.0	9.0	6.0	11.0
	MSE $(\hat{\theta})$	4.0	10.0	5.0	9.0	13.0	1.0	8.0	14.0	2.0	3.0	11.0	15.0	7.0	12.0	6.0
	MSE $(\hat{\lambda})$	14.0	11.0	12.0	7.0	6.0	2.0	10.0	13.0	1.0	8.0	15.0	3.0	5.0	9.0	4.0
	MSE $(\hat{\omega})$	1.0	4.0	8.0	6.0	11.0	14.0	10.0	12.0	2.0	3.0	13.0	15.0	7.0	5.0	9.0
	MRE $(\hat{\theta})$	4.0	9.0	5.0	12.0	10.0	1.0	8.0	14.0	2.0	3.0	11.0	15.0	7.0	13.0	6.0
	MRE $(\hat{\lambda})$	14.0	11.0	12.0	8.0	6.0	3.0	9.0	13.0	1.0	7.0	15.0	5.0	4.0	10.0	2.0
	MRE $(\hat{\omega})$	1.0	2.0	3.0	7.0	10.0	14.0	8.0	12.0	5.0	4.0	13.0	15.0	9.0	6.0	11.0
	D_{abs}	2.0	1.0	6.0	3.0	7.0	14.0	4.0	5.0	13.0	9.0	10.0	15.0	11.0	8.0	12.0
	D_{max}	4.0	2.0	7.0	1.0	5.0	14.0	3.0	9.0	13.0	6.0	11.0	15.0	10.0	8.0	12.0
	$\sum$Ranks	$63.0^{(3)}$	$72.0^{(4)}$	$78.0^{(5)}$	$80.0^{(6)}$	$94.0^{(11)}$	$81.0^{(8)}$	$85.0^{(10)}$	$131.0^{(13)}$	$47.0^{(1)}$	$57.0^{(2)}$	$138.0^{(15)}$	$133.0^{(14)}$	$80.0^{(6)}$	$100.0^{(12)}$	$81.0^{(8)}$
50	$\|Bias\|(\hat{\theta})$	10.0	9.0	8.0	13.0	7.0	1.0	4.0	11.0	2.0	3.0	14.0	15.0	6.0	12.0	5.0
	$\|Bias\|(\hat{\lambda})$	2.0	6.0	11.0	7.0	9.0	13.0	3.0	12.0	1.0	4.0	15.0	14.0	8.0	5.0	10.0
	$\|Bias\|(\hat{\omega})$	4.0	2.0	6.0	10.0	7.0	14.0	1.0	9.0	5.0	3.0	13.0	15.0	12.0	8.0	11.0
	MSE $(\hat{\theta})$	11.0	9.0	8.0	13.0	7.0	1.0	4.0	10.0	2.0	3.0	14.0	15.0	6.0	12.0	5.0
	MSE $(\hat{\lambda})$	10.0	5.0	12.0	8.0	9.0	11.0	2.0	14.0	1.0	6.0	15.0	13.0	3.0	7.0	4.0
	MSE $(\hat{\omega})$	6.0	4.0	5.0	12.0	7.0	14.0	2.0	8.0	1.0	3.0	13.0	15.0	11.0	9.0	10.0
	MRE $(\hat{\theta})$	10.0	9.0	8.0	13.0	7.0	1.0	4.0	11.0	2.0	3.0	14.0	15.0	6.0	12.0	5.0
	MRE $(\hat{\lambda})$	2.0	6.0	11.0	7.0	9.0	13.0	3.0	12.0	1.0	4.0	15.0	14.0	8.0	5.0	10.0
	MRE $(\hat{\omega})$	4.0	2.0	6.0	10.0	7.0	14.0	1.0	9.0	5.0	3.0	13.0	15.0	12.0	8.0	11.0
	D_{abs}	1.0	2.0	6.0	4.0	7.0	14.0	3.0	5.0	11.0	8.0	10.0	15.0	12.0	9.0	13.0
	D_{max}	2.0	1.0	7.0	4.0	6.0	14.0	3.0	5.0	10.0	9.0	11.0	15.0	12.0	8.0	13.0
	$\sum$Ranks	$62.0^{(5)}$	$55.0^{(4)}$	$88.0^{(7)}$	$101.0^{(11)}$	$82.0^{(6)}$	$110.0^{(13)}$	$30.0^{(1)}$	$106.0^{(12)}$	$41.0^{(2)}$	$49.0^{(3)}$	$147.0^{(14)}$	$161.0^{(15)}$	$96.0^{(9)}$	$95.0^{(8)}$	$97.0^{(10)}$
120	$\|Bias\|(\hat{\theta})$	8.0	5.0	6.0	10.0	7.0	1.0	4.0	11.0	2.0	3.0	14.0	15.0	13.0	9.0	12.0
	$\|Bias\|(\hat{\lambda})$	5.0	6.0	10.0	4.0	9.0	13.0	7.0	12.0	1.0	2.0	15.0	14.0	8.0	3.0	11.0
	$\|Bias\|(\hat{\omega})$	1.0	3.0	5.0	4.0	6.0	14.0	2.0	10.0	9.0	8.0	13.0	15.0	11.0	7.0	12.0
	MSE $(\hat{\theta})$	8.0	7.0	4.0	10.0	3.0	1.0	5.0	11.0	2.0	6.0	14.0	15.0	13.0	9.0	12.0
	MSE $(\hat{\lambda})$	3.0	6.0	11.0	4.0	9.0	13.0	7.0	12.0	1.0	2.0	15.0	14.0	8.0	5.0	10.0
	MSE $(\hat{\omega})$	3.0	2.0	6.0	9.0	7.0	14.0	1.0	10.0	4.0	5.0	13.0	15.0	11.0	8.0	12.0
	MRE $(\hat{\theta})$	8.0	5.0	6.0	10.0	7.0	1.0	4.0	11.0	2.0	3.0	14.0	15.0	13.0	9.0	12.0
	MRE $(\hat{\lambda})$	5.0	6.0	10.0	4.0	9.0	13.0	7.0	12.0	1.0	2.0	15.0	14.0	8.0	3.0	11.0
	MRE $(\hat{\omega})$	1.0	3.0	5.0	4.0	6.0	14.0	2.0	10.0	9.0	8.0	13.0	15.0	11.0	7.0	12.0
	D_{abs}	2.0	4.0	6.0	1.0	8.0	14.0	5.0	7.0	10.0	9.0	11.0	15.0	13.0	3.0	12.0
	D_{max}	2.0	3.0	6.0	1.0	7.0	14.0	4.0	8.0	10.0	9.0	13.0	15.0	11.0	5.0	12.0
	$\sum$Ranks	$46.0^{(1)}$	$50.0^{(3)}$	$75.0^{(8)}$	$61.0^{(6)}$	$78.0^{(9)}$	$112.0^{(10)}$	$48.0^{(2)}$	$114.0^{(11)}$	$51.0^{(4)}$	$57.0^{(5)}$	$150.0^{(14)}$	$162.0^{(15)}$	$120.0^{(12)}$	$68.0^{(7)}$	$128.0^{(13)}$
180	$\|Bias\|(\hat{\theta})$	2.0	6.0	11.0	3.0	10.0	1.0	7.0	9.0	4.0	8.0	14.0	15.0	13.0	5.0	12.0
	$\|Bias\|(\hat{\lambda})$	2.0	6.0	11.0	1.0	10.0	13.0	7.0	12.0	3.0	5.0	15.0	14.0	8.0	4.0	9.0
	$\|Bias\|(\hat{\omega})$	1.0	2.0	7.0	3.0	8.0	14.0	4.0	9.0	10.0	6.0	13.0	15.0	11.0	5.0	12.0
	MSE $(\hat{\theta})$	2.0	5.0	11.0	3.0	10.0	1.0	6.0	9.0	4.0	8.0	14.0	15.0	13.0	7.0	12.0
	MSE $(\hat{\lambda})$	3.0	6.0	11.0	1.0	10.0	13.0	7.0	12.0	4.0	5.0	15.0	14.0	8.0	2.0	9.0
	MSE $(\hat{\omega})$	1.0	2.0	10.0	3.0	9.0	14.0	4.0	5.0	7.0	8.0	13.0	15.0	11.0	6.0	12.0
	MRE $(\hat{\theta})$	2.0	6.0	11.0	3.0	10.0	1.0	7.0	9.0	4.0	8.0	14.0	15.0	13.0	5.0	12.0
	MRE $(\hat{\lambda})$	2.0	6.0	11.0	1.0	10.0	13.0	7.0	12.0	3.0	5.0	15.0	14.0	8.0	4.0	9.0
	MRE $(\hat{\omega})$	1.0	2.0	7.0	3.0	8.0	14.0	4.0	9.0	10.0	6.0	13.0	15.0	11.0	5.0	12.0
	D_{abs}	1.0	3.0	7.0	4.0	6.0	14.0	2.0	8.0	11.0	9.0	10.0	15.0	13.0	5.0	12.0
	D_{max}	2.0	3.0	7.0	4.0	6.0	14.0	1.0	8.0	10.0	9.0	13.0	15.0	12.0	5.0	11.0
	$\sum$Ranks	$19.0^{(1)}$	$47.0^{(3)}$	$104.0^{(10)}$	$29.0^{(2)}$	$97.0^{(8)}$	$112.0^{(11)}$	$56.0^{(5)}$	$102.0^{(9)}$	$70.0^{(6)}$	$77.0^{(7)}$	$149.0^{(14)}$	$162.0^{(15)}$	$121.0^{(12)}$	$53.0^{(4)}$	$122.0^{(13)}$
250	$\|Bias\|(\hat{\theta})$	3.0	5.0	8.0	4.0	9.0	1.0	6.0	12.0	2.0	7.0	14.0	15.0	10.0	13.0	11.0
	$\|Bias\|(\hat{\lambda})$	2.0	6.0	11.0	3.0	10.0	13.0	7.0	12.0	1.0	4.0	15.0	14.0	8.0	5.0	9.0
	$\|Bias\|(\hat{\omega})$	1.0	4.0	8.0	2.0	7.0	14.0	3.0	6.0	5.0	9.0	13.0	15.0	12.0	10.0	11.0
	MSE $(\hat{\theta})$	3.0	5.0	8.0	4.0	9.0	1.0	6.0	10.0	2.0	7.0	14.0	15.0	11.0	13.0	12.0
	MSE $(\hat{\lambda})$	2.0	5.0	11.0	3.0	10.0	13.0	6.0	12.0	1.0	4.0	15.0	14.0	8.0	7.0	9.0
	MSE $(\hat{\omega})$	1.0	3.0	8.0	4.0	7.0	14.0	2.0	6.0	5.0	9.0	13.0	15.0	11.0	12.0	10.0
	MRE $(\hat{\theta})$	3.0	5.0	8.0	4.0	9.0	1.0	6.0	12.0	2.0	7.0	14.0	15.0	10.0	13.0	11.0
	MRE $(\hat{\lambda})$	2.0	6.0	11.0	3.0	10.0	13.0	7.0	12.0	1.0	4.0	15.0	14.0	8.0	5.0	9.0
	MRE $(\hat{\omega})$	1.0	4.0	8.0	2.0	7.0	14.0	3.0	6.0	5.0	9.0	13.0	15.0	12.0	10.0	11.0
	D_{abs}	1.0	4.0	5.0	2.0	7.0	14.0	3.0	6.0	10.0	9.0	11.0	15.0	12.0	8.0	13.0
	D_{max}	2.0	4.0	6.0	1.0	5.0	14.0	3.0	7.0	10.0	9.0	13.0	15.0	11.0	8.0	12.0
	$\sum$Ranks	$21.0^{(1)}$	$51.0^{(4)}$	$92.0^{(8)}$	$32.0^{(2)}$	$90.0^{(7)}$	$112.0^{(11)}$	$52.0^{(5)}$	$101.0^{(9)}$	$44.0^{(3)}$	$78.0^{(6)}$	$150.0^{(14)}$	$162.0^{(15)}$	$113.0^{(12)}$	$104.0^{(10)}$	$118.0^{(13)}$
350	$\|Bias\|(\hat{\theta})$	2.0	8.0	11.0	3.0	12.0	1.0	5.0	13.0	4.0	7.0	14.0	15.0	10.0	6.0	9.0
	$\|Bias\|(\hat{\lambda})$	1.0	7.0	10.0	2.0	11.0	13.0	6.0	12.0	4.0	5.0	15.0	14.0	8.0	3.0	9.0
	$\|Bias\|(\hat{\omega})$	1.0	3.0	8.0	2.0	10.0	14.0	5.0	7.0	9.0	6.0	13.0	15.0	12.0	4.0	11.0
	MSE $(\hat{\theta})$	2.0	4.0	11.0	3.0	13.0	1.0	6.0	12.0	5.0	7.0	14.0	15.0	9.0	8.0	10.0
	MSE $(\hat{\lambda})$	2.0	6.0	11.0	1.0	10.0	13.0	5.0	12.0	4.0	7.0	15.0	14.0	8.0	3.0	9.0
	MSE $(\hat{\omega})$	1.0	3.0	9.0	2.0	10.0	14.0	4.0	7.0	8.0	6.0	13.0	15.0	12.0	5.0	11.0
	MRE $(\hat{\theta})$	2.0	8.0	11.0	3.0	12.0	1.0	5.0	13.0	4.0	7.0	14.0	15.0	10.0	6.0	9.0
	MRE $(\hat{\lambda})$	1.0	7.0	10.0	2.0	11.0	13.0	6.0	12.0	4.0	5.0	15.0	14.0	8.0	3.0	9.0
	MRE $(\hat{\omega})$	1.0	3.0	8.0	2.0	10.0	14.0	5.0	7.0	9.0	6.0	13.0	15.0	12.0	4.0	11.0
	D_{abs}	1.0	3.0	5.0	2.0	8.0	14.0	4.0	6.0	10.0	9.0	11.0	15.0	12.0	7.0	13.0
	D_{max}	1.0	3.0	6.0	2.0	8.0	14.0	4.0	9.0	10.0	7.0	13.0	15.0	12.0	5.0	11.0
	$\sum$Ranks	$15.0^{(1)}$	$55.0^{(4)}$	$100.0^{(8)}$	$24.0^{(2)}$	$115.0^{(13)}$	$112.0^{(10)}$	$55.0^{(4)}$	$110.0^{(9)}$	$71.0^{(6)}$	$72.0^{(7)}$	$150.0^{(14)}$	$162.0^{(15)}$	$113.0^{(12)}$	$54.0^{(3)}$	$112.0^{(10)}$

Table VII contains numerical results from the simulation study of the ARS-W model, for parameters $\theta = 0.25$, $\lambda = 1.75$, and $\omega = 0.5$, show a general trend of improving estimator performance as the sample size (n) increases from 15 to 350, evidenced by decreasing values for most metrics. The MSADE and the MSALDE consistently rank among the best methods for small sample sizes ($n = 15$), with MSADE having the lowest overall cumulative rank. However, as the sample size increases, the MLE emerges as the overall superior method, achieving the best cumulative rank for $n = 180$, $n = 250$, and $n = 350$. This suggests that for large samples, the MLE's theoretical efficiency is realized. The RTADE is the best individual estimator for the θ parameter, consistently achieving rank 1 for its absolute bias and MSE across all sample sizes. However, its extremely poor performance in estimating the other parameters and for the overall distance metrics (D_{abs} and D_{max}) results in it having a poor overall rank, particularly at larger sample sizes. The MPSE and the WLSE also perform very well, with the MPSE being the second best overall at $n = 180$ and $n = 250$, and the WLSE being the best overall at $n = 50$ and the second best at $n = 120$. Conversely, the ADSOE and the KE are consistently the worst performers across all metrics and sample sizes, demonstrating the highest bias, MSE, and MRE values, and the largest overall distance measures. Their overall ranks are consistently 14 and 15. The difference between the best and worst methods is substantial, particularly for the θ parameter and for the overall MSE metric.
Table VIII displays the partial and overall ranks for sixteen different estimation methods of the ARS-W model across several sample sizes (n), where rank 1 indicates the best performance. The MSADE is the best overall method for the smallest sample size ($n = 15$), followed closely by the MSALDE. As the sample size increases to $n = 120$ and beyond, the MLE becomes the superior method, achieving the best overall cumulative rank at $n = 120$, $n = 180$, $n = 250$, and $n = 350$. The MPSE is a very strong competitor to the MLE, securing the second-best overall rank for the larger sample sizes. The WLSE is also highly effective, particularly at $n = 50$ and $n = 120$, where it ranks first and second overall, respectively. The RTADE consistently shows the best performance for the $\hat{\theta}$ parameter estimate across all sample sizes, but its high rank in other metrics results in a low overall rank, typically in the range of 8 to 11. Conversely, the ADSOE and the KE consistently rank as the two worst methods, performing poorly for almost all metrics across all sample sizes. In general, the performance of the best estimators improves relative to the others as the sample size increases, confirming the desirable asymptotic properties of the most effective methods. Figures 13, 15, 14 and 16 are plots for the Bias, MSE, MRE, D_{abs} and D_{max} which support the results in Table VII.

7. APPLICATIONS

In this section, we apply the ARS-W model to three datasets so as to determine its potential to fit real-life data. The first data, here referred to as Data-I contained in Table (IX) represent the values of the survival times given in days of guinea pigs infected with virulent tubercle bacilli obtained by Bjerkedal [74] and analyzed in Shama et al. [75].

TABLE IX. Data-I

0.1	0.74	1	1.08	1.16	1.3	1.53	1.71	1.97	2.3	2.54	3.47
0.33	0.77	1.02	1.08	1.2	1.34	1.59	1.72	2.02	2.31	2.54	3.61
0.44	0.92	1.05	1.09	1.21	1.36	1.6	1.76	2.13	2.4	2.78	4.02
0.56	0.93	1.07	1.12	1.22	1.39	1.63	1.83	2.15	2.45	2.93	4.32
0.59	0.96	1.07	1.13	1.22	1.44	1.63	1.95	2.16	2.51	3.27	4.58
0.72	1	1.08	1.15	1.24	1.46	1.68	1.96	2.22	2.53	3.42	5.55

Data-II represent the active repair times (hours) for an airborne communication transceiver studied by Yousof et al. [76] and El-Saeed, Obulezi, and Abd El-Raouf [77]. This data is presented in Table (X).

TABLE X. Data-II

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8	1.0	1.0	1.0
1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3
4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3	22.0	24		

Table (XI) shows 40 observations representing time to failure (103h) of the turbocharger from one type of engine studied by Xu et al. [78]. This data is referred subsequently as Data-III.

TABLE XI. Data-III

7	1.6	7.7	6.5	2.0	7.1	8.3	2.6	8.1	3.0	6.7	8.4	7.8	3.5
7.9	8.4	8.5	3.9	8.7	4.5	8.8	4.6	9.0	6.0	4.8	8.0	5.0	7.3
5.1	7.3	5.3	7.3	5.4	6.1	5.6	6.0	5.8	6.3	7.0	6.5		

TABLE XII. Summmary Statistics for Data-I, Data-II and Data-III

Statistics	Data-I	Data-II	Data-III
n	72	46	40
Q_1	1.08	0.8	5.075
Q_3	2.24	4.375	7.825
IQR	1.16	3.575	2.75
Outliers	4.02, 4.43, 4.58, 5.55	10.3, 22, 24	–
Mean	1.768194	3.595652	6.2350
Median	1.495	1.75	6.50
Variance	1.070291	23.9862	3.784385
Std. Dev.	1.034549	4.897571	1.945349
Range	5.45	23.8	7.40
Skewness	1.341869	2.856083	-0.66488
Kurtosis	4.991056	11.58927	2.66488

The summary statistics of Data-I, Data-II, and Data-III presented in Table (XII) reveal distinctive characteristics for each dataset. Data-I with 72 observations is very compact with a mean of 1.768 and median of 1.495, revealing a weak positive skewness as revealed by the measure of skewness being 1.341869. Its 1.16 interquartile range (IQR) is lowest of the three, and it has a standard deviation of 1.034549, suggesting less spread than in the other datasets but with the suggestion of the existence of outliers at 4.02, 4.43, 4.58, and 5.55 that some of the higher values are skewing the distribution. The kurtosis of 4.991056 suggests a reasonably heavy-tailed distribution compared to a normal distribution. Data-II, with 46 observations, has a much greater spread as evidenced from its range of 23.8 and a much greater standard deviation of 4.897571 and variance of 23.9862. Its mean of 3.595652 is quite different from its median of 1.75, which, combined with a large skewness of 2.856083, strongly proves to be a heavy right (positive) skew. This positive skew is once more pointed out by the presence of the extreme values of 10.3, 22, and 24, which are several times greater than the third quartile (Q_3) of 4.375. The very high kurtosis of 11.58927 reflects a highly peaked distribution with very heavy tails, likely the result of these extreme values. Data-III with 40 data points follows a different trend. Its mean is 6.2350 and median is 6.50, very close to each other, but the negative skewness of -0.66488 indicates a slightly left (negative) skew, meaning the tail of the distribution leans more towards lower values. There being no listed outliers means that the dataset is more bounded within its assumed range. Its spread is between Data-I and Data-II, with an IQR of 2.75 and standard deviation of 1.945349. The kurtosis of 2.66488 is below 3, the Platykurtic point, hence a platykurtic distribution indicating it is less peaked and has lighter tails as compared to a normal distribution. In general, however, Data-I and Data-II both have positive skewness and the presence of outliers, Data-II is characterized by the extreme spread and high positive skewness, whereas Data-III shows comparatively more symmetric, or slightly negatively skewed, distribution without the presence of extreme outliers.

Some well-known distributions were chosen to compare with the ARS-W distribution and demonstrate its goodness of fit. These distributions are the flexible Weibull (FW) distribution by [79], Gumbel distribution by [80], Burr XII distribution by [81], Gamma distribution, Lognormal distribution, Weibull (W) distribution by [51], the logistic cotangent Weibull (LCW) distribution by [82]. We evaluate the performance of each fitted distribution using several metrics: log-likelihood (LL), Akaike Information Criterion (AIC), Consistent AIC (CAIC), Bayesian Information Criterion (BIC), and Hannan–Quinn Information Criterion (HQIC). To assess goodness-of-fit, we also look at the Cramér–von Mises statistic (W), Anderson–Darling statistic (A), Kolmogorov–Smirnov (K-S) statistic, and their corresponding p-value.

TABLE XIII. Model Comparison and Adequacy for Data-I, Data-II and Data-III

Data	Distr.	LL	AIC	CAIC	BIC	HQIC	W	A	KS	p-value
Data-I	ARS-W	-93.08	192.1446	192.4976	198.9746	194.8637	0.0633	0.4034	0.0769	0.7878
	FW	-102.43	208.867	209.0409	213.4204	210.6797	0.2583	1.6149	0.1464	0.0912
	Gumbel	-93.51	191.0245	191.1984	195.5778	192.8371	0.0954	0.5397	0.0901	0.6030
	Gamma	-94.23	192.4582	192.6321	197.0115	194.2709	0.0971	0.5985	0.0907	0.5946
	Lognormal	-97.52	199.0443	199.2182	203.5976	200.857	0.1074	0.7713	0.1104	0.3435
	Burr XII	-98.42	200.8386	201.0125	205.392	202.6513	0.1193	0.8992	0.1294	0.1791
	Weibull	-95.79	195.5796	195.7535	200.1329	197.3923	0.1649	0.9710	0.1048	0.4075
	LCW	-115.7	235.3935	235.5674	239.9468	237.2062	0.6273	3.5739	0.2354	0.0007
Data-II	ARS-W	-100.8209	207.6118	208.1832	213.0977	209.6669	0.0702	0.4401	0.0923	0.8277
	FW	-101.43	206.855	207.1341	210.5123	208.225	0.0544	0.4964	0.1507	0.2468
	Gumbel	-117.9	239.7992	240.0783	243.4563	241.1693	0.3716	2.3233	0.1985	0.0533
	Gamma	-104.8	213.5998	213.8789	217.2571	214.9698	0.1425	0.9888	0.1454	0.2849
	Lognormal	-99.95	203.9054	204.1844	207.5627	205.2754	0.0523	0.3315	0.0946	0.8052
	Burr XII	-101.06	206.1213	206.4004	209.7786	207.4913	0.0759	0.4496	0.0937	0.8139
	Weibull	-104.36	212.7174	212.9964	216.3746	214.0874	0.1295	0.8984	0.1200	0.5222
	LCW	-106.97	217.9489	218.2280	221.6062	219.3189	0.1056	0.9039	0.2693	0.0025
Data-III	ARS-W	-81.2836	168.3729	169.0395	173.4395	170.2048	0.0411	0.3316	0.0724	0.9848
	FW	-83.77	171.5468	171.8711	174.9246	172.7681	0.1100	0.7895	0.1291	0.5180
	Gumbel	-88.78	181.5556	181.8799	184.9334	182.7769	0.2307	1.4998	0.1496	0.3323
	Gamma	-87.18	178.3696	178.6939	181.7473	179.5909	0.2008	1.3286	0.1306	0.5028
	Lognormal	-90.61	185.2176	185.5419	188.5953	186.4389	0.3010	1.8898	0.1523	0.3118
	Burr XII	-133.29	270.5844	270.9087	273.9621	271.8056	0.5206	3.0382	0.4236	1.169×10^{-6}
	Weibull	-82.28	168.5596	168.8839	171.9374	169.7809	0.0694	0.5268	0.1028	0.7922
	LCW	-85.57	175.1477	175.472	178.5255	176.369	0.1086	0.7758	0.2727	0.0052

Table (XIII) provides a comprehensive comparison of various probability distributions that have been fit to Data-I, Data-II, and Data-III, comparing their fit on the basis of log-likelihood, information criteria, and goodness-of-fit tests. For Data-I, the best among all is the Gumbel distribution with the best log-likelihood (-93.51) and the lowest values across all information criteria (AIC, CAIC, BIC, HQIC). Critically, with a KS p-value of 0.6030 at 0.6030 far beyond typical significance levels, the fit is good to the data. The ARS-W distribution also has very high p-value performance at 0.7878 but the information criteria are slightly poorer than Gumbel. By comparison, LCW distribution clearly does not work well with Data-I, as shown by its very low log-likelihood value, high information criteria, and very low p-value of 0.0007 strongly rejecting its fitness. Moving to Data-II, the best fitted model is the Lognormal distribution. It has the optimum log-likelihood value of -99.95 and also the lowest for AIC, CAIC, BIC, and HQIC, indicating the best balance between model fit and parsimony. Besides, its KS p-value of 0.8052 strongly supports its fitness in modeling Data-II. ARS-W and Burr XII distributions also show good fits with good p-values (0.8277 and 0.8139, respectively), but their information criteria are slightly less optimal than the Log-normal. Gumbel distribution, on the other hand, is evident to be not a good model for Data-II by its very low log-likelihood, high information criteria, and marginal KS p-value of 0.0533, which makes it doubtful for this data. Lastly, in Data-III, ARS-W distribution exhibits the best fit. It has the highest log-likelihood of -81.2836 and lowest information criteria values compared to all considered models. Most significantly, its extremely high KS p-value of 0.9848 provides strong evidence that Data-III closely follows the ARS-W distribution. Weibull distribution also provides a good fit with comparable information criteria and an extremely large p-value of 0.7922 and is thus a very strong alternative, albeit not quite as good as ARS-W. The Burr XII distribution, though, is firmly not appropriate for Data-III, which is defined by its extremely low log-likelihood, excessively large information criteria, and a very small p-value of 1.169×10^{-6}, indicating a bad fit. Briefly, employing joint evaluation of log-likelihood, information criteria, and goodness-of-fit p-values, the Gumbel distribution would be most suitable for Data-I, the Log-normal distribution would be best suited for Data-II, and the ARS-W distribution would be most well-fitted to Data-III.

TABLE XIV. Maximum Likelihood Estimates and (Standard Errors) for Data-I, Data-II, and Data-III

Data	Distr.	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\omega}$
Data-I	ARS-W	0.1207(0.1019)	0.0111(0.0095)	2.7805(0.3041)
	FW	0.3824(0.0367)	1.4381(0.1781)	–
	Gumbel	1.3229(0.0911)	0.7368(0.0693)	–
	Gamma	3.0836(0.4887)	1.7439(0.3001)	–
	Lognormal	0.3991(0.0741)	0.6291(0.0524)	–
	Burr XII	3.8127(0.5439)	0.4783(0.0792)	–
	Weibull	1.9959(0.1363)	1.8252(0.1587)	–
	LCW	0.4250(0.0519)	1.1629(0.0678)	–
Data-II	ARS-W	0.0522(0.0430)	0.0063(0.0050)	1.4848(0.1672)
	FW	0.0717(0.0113)	1.1307(0.1822)	–
	Gumbel	1.9199(0.3512)	2.3135(0.3031)	–
	Gamma	0.9359(0.1708)	0.2602(0.0619)	–
	Lognormal	0.6580(0.1623)	1.1007(0.1147)	–
	Burr XII	2.0749(0.3581)	0.5099(0.1029)	–
	Weibull	3.3871(0.5886)	0.9009(0.0961)	–
	LCW	0.2517(0.0356)	0.7625(0.0477)	–
Data-III	ARS-W	8.0254(4.2209)	0.0052(0.0021)	2.7564(0.1765)
	FW	0.2593(0.0333)	11.8110(1.7957)	–
	Gumbel	5.2202(0.3494)	2.0769(0.2336)	–
	Gamma	7.7716(1.7018)	1.2465(0.2820)	–
	Lognormal	1.7645(0.0631)	0.3993(0.0446)	–
	Burr XII	22.6115(59.8990)	0.0251(0.0663)	–
	Weibull	6.8994(0.2936)	3.8762(0.5168)	–
	LCW	0.0180(0.0067)	2.2340(0.1791)	–

Table (XIV) presents the maximum likelihood estimates (MLE) and the corresponding standard errors for the parameters of various distributions that have been fitted to Data-I, Data-II, and Data-III. These estimated parameters are used to define the specific form of each fitted distribution. For Data-I, the Gumbel distribution, which was a good fit in the previous analysis, has parameter estimates $\hat{\theta} = 1.3229$ with standard error 0.0911 and $\hat{\lambda} = 0.7368$ with standard error 0.0693. The relatively low standard errors confirm that the Gumbel distribution parameter estimates are very close to their true values. These other distributions, such as the FW, Gamma, Lognormal, and Weibull, also have reasonably good estimates with very small standard errors relative to their corresponding parameter values. The three-parameter ARS-W distribution has an estimated $\hat{\omega} = 2.7805$ with standard error 0.3041, along with its other two parameters. In Data II, for which the Lognormal distribution was the best appropriate, its parameters are estimated as $\hat{\theta} = 0.6580$ with standard error 0.1623 and $\hat{\lambda} = 1.1007$ with standard error 0.1147. Such standard errors are very small, reflecting exact parameter estimates for the Log-normal distribution in the case of Data-II. The FW distribution also provides quite exact estimates of its parameters. On the other hand, the Gumbel distribution to which Data-II did not have a good fit has greater standard errors than its estimates (e.g., for $\hat{\theta} = 1.9199$ with standard error equal to 0.3512) and thus conveys less confidence in these parameter estimates, in line with its lower model adequacy. In Data-III, the best-fit model is the ARS-W distribution with estimated parameter $\hat{\theta} = 8.0254$ and large standard error of 4.2209, $\hat{\lambda} = 0.0052$ and standard error of 0.0021, and $\hat{\omega} = 2.7564$ with standard error of 0.1765. Although $\hat{\theta}$ has a large standard error, the other two parameters are better estimated. The Weibull distribution, being an appropriate model for Data-III as well, gives us estimates $\hat{\theta} = 6.8994$ (standard error 0.2936) and $\hat{\lambda} = 3.8762$ (standard error 0.5168) which are fairly accurate parameter estimations. Surprisingly, the Burr XII distribution that lacked a satisfactory fit to Data-III has extremely large standard errors for its parameters (e.g. $\hat{\theta} = 22.6115$ with a standard error of 59.8990 and $\hat{\lambda} = 0.0251$ with a standard error of 0.0663), which indicate the unreliability of its parameter estimates due to the poor fit. Overall, the goodness of the parameter estimates will generally be consistent with the goodness of the individual distributions to the individual datasets, where more accurate models will generally have more accurate parameter values.

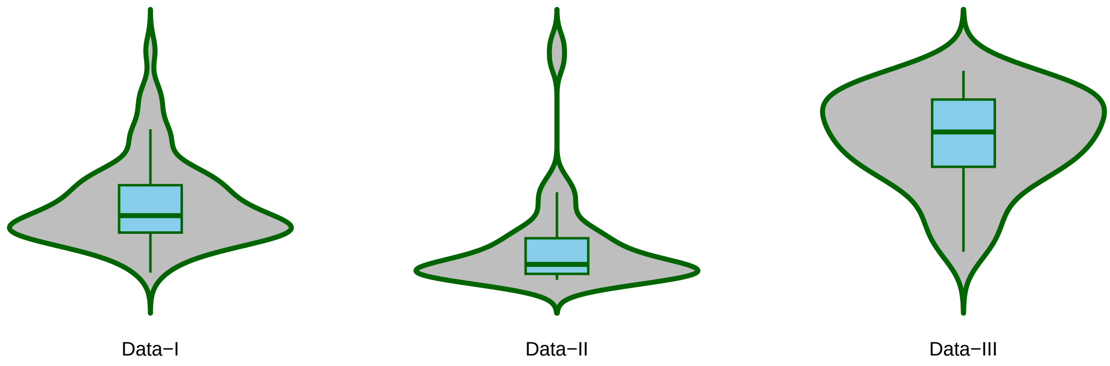

Fig. 17. Boxplot superimposed on Violin plot for the Datasets

Figure (17) vividly illustrate the distinct distributional characteristics of Data-I, Data-II, and Data-III. Data-I presents a distribution that is somewhat right-skewed, indicated by the longer upper tail of its violin shape and the median line being slightly closer to the bottom of the box, though it maintains a relatively symmetrical central body. Data-II, in stark contrast, exhibits a highly right-skewed distribution with a much wider spread, particularly evident in its elongated upper tail and the boxplot's median positioned significantly lower within the box, reflecting the presence of higher values and greater variability. Conversely, Data-III displays a distribution that is left-skewed and notably more concentrated, as shown by its narrower, more compact violin shape and the median line positioned higher within the box, indicating a clustering of data points towards the higher end of its range and a less dispersed nature compared to the other two datasets.

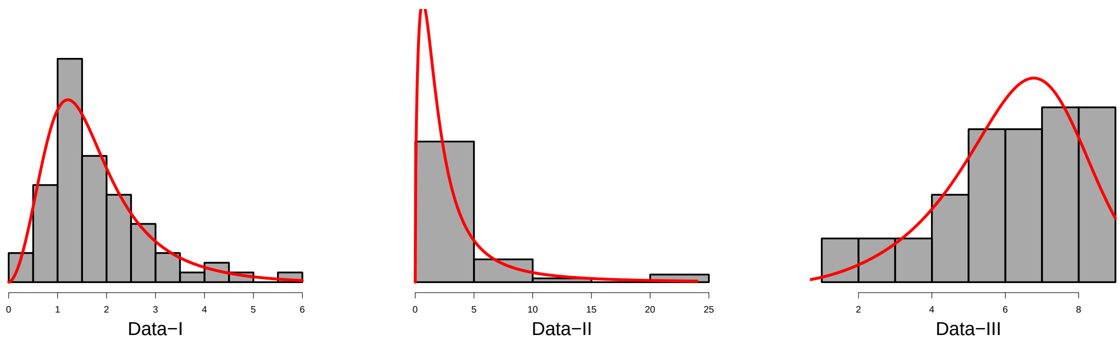

Fig. 18. Density plot superimposed on Histogram for the Datasets

Figure (18) offer a visual confirmation of the distributional characteristics previously discussed for Data-I, Data-II, and Data-III. Data-I clearly exhibits a right-skewed distribution, with the majority of observations clustered at lower values and a tail extending towards higher values, as indicated by both the histogram bars and the fitted red density curve. Data-II presents an even more pronounced right-skewness, with a very high frequency at the lowest values and a long, sparse tail, underscoring its extreme variability. In contrast, Data-III demonstrates a left-skewed distribution, with the bulk of the data concentrated at higher values and a tail extending towards lower values, further confirming its more compact and tightly grouped nature.

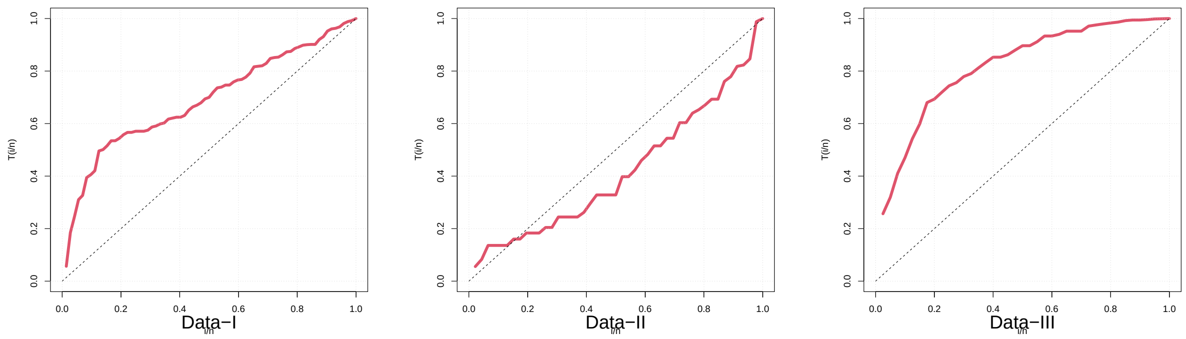

Fig. 19. TTT plots for the Datasets

The TTT plots of Data-I, Data-II, and Data-III in Figure (19) give valuable visual clues regarding their underlying hazard rate behaviors. For Data-I, the curve is largely above the diagonal, particularly after the initial portion, and appears to be concave, a strong suggestion of an increasing hazard rate over the majority of its range. Conversely, the TTT plot of Data-II, in which the curve is consistently below the diagonal and convex in nature, clearly suggests a decreasing hazard rate, i.e., the probability of an event occurring diminishes as time passes. In the meantime, the TTT plot of Data-III is a curve that is also mostly above the diagonal and is prominently concave, especially in its early to mid-stages, which indicates an increasing hazard rate, meaning that the chance of an event is growing over time. These different visual patterns in the TTT plots give a qualitative analysis of the underlying hazard functions of each dataset.

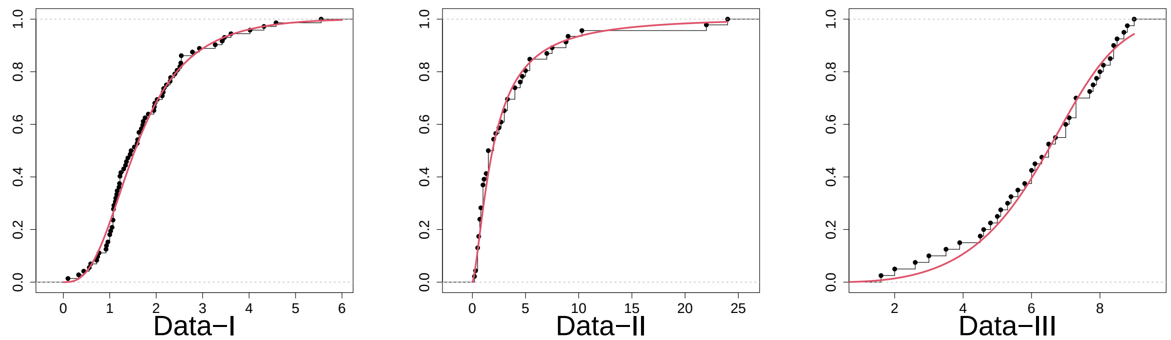

Fig. 20. Empirical versus theoretical CDF plots for the Datasets

The empirical versus theoretical CDF plots in Figure (20) provide a visual assessment of how well a theoretical distribution fits the observed data for Data-I, Data-II, and Data-III. For Data-I, the empirical CDF (black points) closely follows the theoretical CDF (red line), particularly in the central portion of the distribution, suggesting a good overall fit. Similarly, Data-II also shows a strong alignment between its empirical and theoretical CDFs, indicating that the chosen theoretical distribution effectively captures the cumulative probabilities of this dataset, despite its wider spread. Data-III, too, demonstrates a close correspondence between its empirical and theoretical CDFs, with the black points generally hugging the red curve, which implies a satisfactory fit of the theoretical model to the observed data's cumulative distribution. Overall, these plots visually reinforce the adequacy of the theoretical models in representing the cumulative probability patterns across all three datasets.

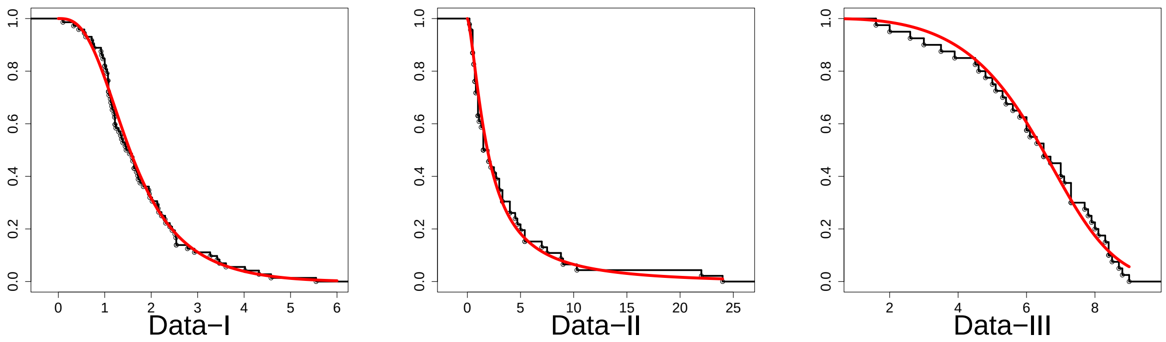

Fig. 21. Empirical versus theoretical survival function plots for the Datasets

The empirical versus theoretical survival function plots for Data-I, Data-II, and Data-III in Figure (21) offer a visual assessment of how well the theoretical distribution models the survival probabilities of the observed data. For Data-I, the empirical survival function (black stepped line) closely tracks the theoretical survival function (red smooth curve), particularly across the range where most events occur, indicating a good fit of the theoretical model to the observed survival patterns. Similarly, Data-II demonstrates a strong concordance between its empirical and theoretical survival functions, with the red curve accurately mirroring the stepped black line, suggesting that the model effectively captures the survival probabilities for this dataset, even with its extended tail. Data-III also shows a close alignment, where the empirical survival function generally follows the theoretical curve, reinforcing the conclusion that the theoretical model provides a satisfactory representation of the survival characteristics for all three datasets.

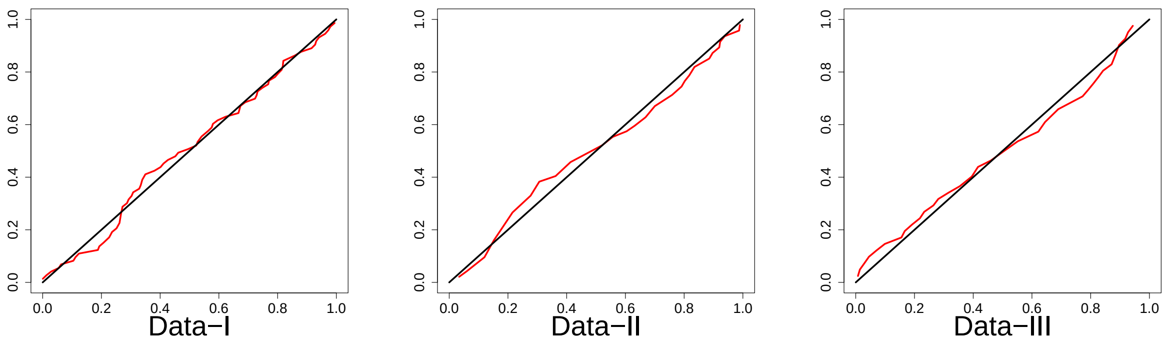

Fig. 22. P-P plots for the Datasets

The P-P (Probability-Probability) plots of Data-I, Data-II, and Data-III in Figure (22) present a graphical analysis of the goodness of fit of the probabilities of a given theoretical distribution with empirical probabilities of observed data. In Data-I, the plot well captures the red line, representing the empirical probabilities, closely tracking the diagonal black line. This high level of correspondence indicates an excellent fit, suggesting that the theoretical distribution extremely well captures the observed probability distribution of Data-I. In contrast, Data-II's P-P plot is more skewed. Empirical probabilities track down the diagonal overall, but the red line is unmistakably shifted into an S-shape, veering away from the diagonal in the middle range before again falling back to it at the extremes. This suggests that the theoretical model, as fine as it is, likely fails to exactly capture Data-II's probability structure, so there is marginally less perfect fit than for Data-I. In Data-III, the P-P plot demonstrates a mostly close consensus between theoretical and empirical probabilities. The red line is mostly close to the ideal line, which validates the conclusion that the theoretical distribution produces a good but not an ideal fit to the probability characteristics of this data, with some oscillations around the ideal line. In general, the P-P plots provide a visual confirmation of the differential levels of adequacy of the chosen theoretical model in explaining the probability distributions among the three alternative datasets.

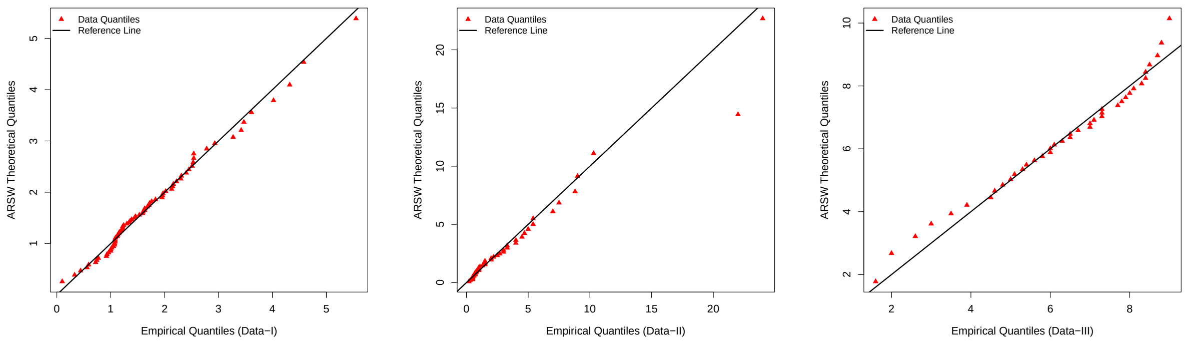

Fig. 23. Q-Q plots for the Datasets

The Q-Q (Quantile-Quantile) plots of Data-I, Data-II, and Data-III in Figure (23) provide a qualitative assessment of the goodness of fit between the theoretical quantiles of the ARS-W distribution and the empirical quantiles of the three datasets. For Data-I, the red dots, i.e., the quantiles of data, mostly occur very close to the black reference line of 45 degrees, indicating a high consistency of the empirical distribution of Data-I with the theoretical distribution ARS-W. It suggests a good fit for most of the range of the data. However, in the case of Data-II, though the quantiles of the data trace the reference line in most of the distribution, there is a dramatic deviation in the upper tail. The red points clearly stray from the diagonal for larger empirical quantile values, which signifies that the ARS-W distribution does not describe the behavior of extreme values of Data-II. It signifies a poorer fit for Data-II, particularly in the right tail. On the other hand, Data-III's Q-Q plot reveals a very close correspondence between its empirical and theoretical quantiles with the red dots hugging the diagonal line throughout. This level of correspondence confirms the conclusion that the ARS-W distribution is a very good and highly satisfactory fit with the quantile performance of the observed data for Data-III. Cumulatively, these plots offer pictorial evidence of the ARS-W model's strong adequacy to Data-I and Data-III, but also record its deficiency in proper characterization of the upper tail of Data-II.

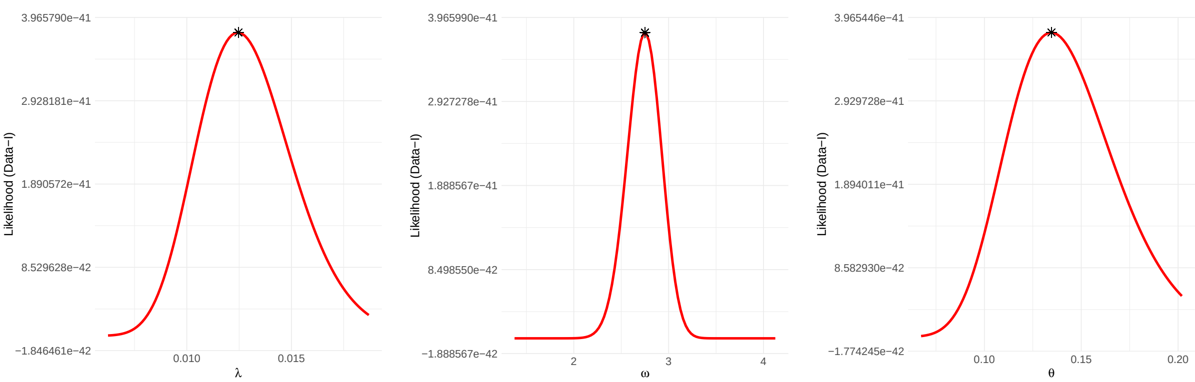

Fig. 24. Likelihood Profile for the Data-I

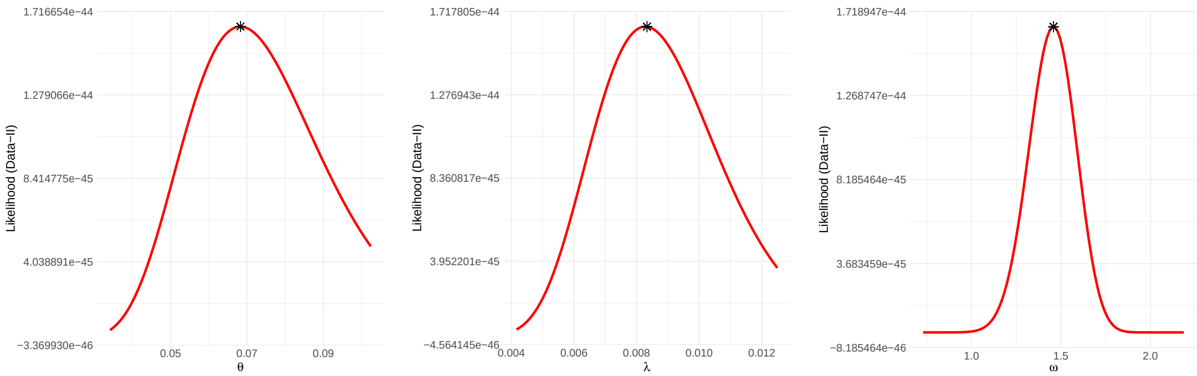

Fig. 25. Likelihood Profile for the Data-II

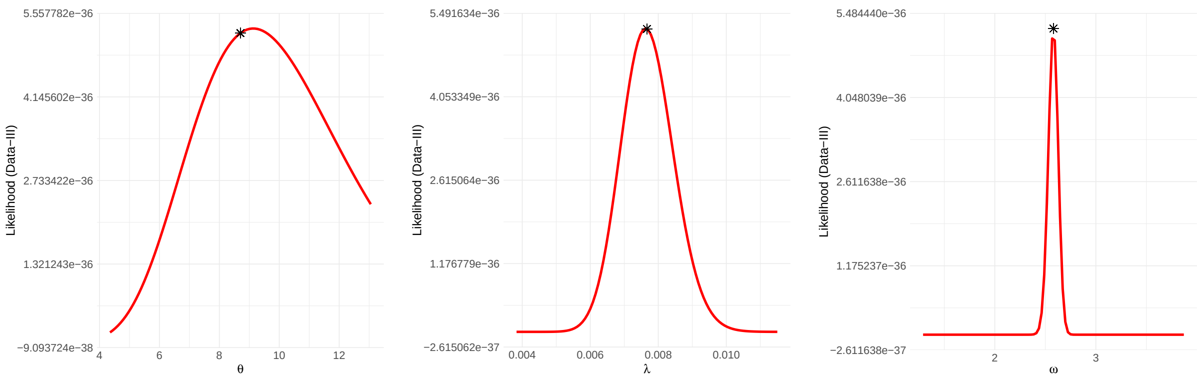

Fig. 26. Likelihood Profile for the Data-III

The likelihood profile plots for Data-I, Data-II, and Data-III in Figures (24-26) provide a visual representation of how sensitive the likelihood function is to changes in each parameter, with the black star indicating the Maximum Likelihood Estimate (MLE). For Data-I, the profiles for all three parameters (θ, λ, and ω) show well-defined, unimodal peaks, suggesting that the MLEs are unique and the likelihood function is well-behaved around these optimal points. Similarly, Data-II's likelihood profiles also exhibit distinct peaks, although the curves appear somewhat narrower, implying that the likelihood function is more sharply peaked around the MLEs for this dataset. Data-III's profiles, particularly for θ and λ, display very sharp and narrow peaks, indicating a high degree of precision in the estimation of these parameters, which aligns with the low variability observed in Data-III. Across all datasets, the consistent presence of clear peaks in the likelihood profiles supports the robustness of the parameter estimation process for the fitted distribution.

8. FINAL REMARKS AND FUTURE WORK

This paper introduces the ARS family of distributions, with a specific focus on the ARS-W distribution—a new trigonometric model with superior data adaptability. We explore the family's structural features to demonstrate its practical benefits.To evaluate the model's parameter precision and robustness, we conducted a Monte Carlo simulation using seven traditional estimation methods: maximum likelihood (ML), maximum product of spacing (MPS), least squares (LS), weighted least squares (WLS), Cramér-von Mises (CVM), Anderson-Darling (AD), and right-tail Anderson-Darling (RTAD). We further validated its effectiveness by applying it to complex lifetime datasets namely survival times of guinea pigs infected with virulent tubercle bacilli, active repair times for an airborne communication transceiver and time to failure of turbocharger from one type of engine. The results show the ARS-W model to be a powerful and reliable tool for complex tasks in multiple disciplines. However, its flexibility is reduced when the underlying data exhibits unimodal, bathtub, or reversed bathtub shapes.

The large bias values for the parameter ω in the estimation tables are a significant concern, as they indicate a substantial and systematic deviation between the estimated values and the true parameter value. This suggests the estimators are not performing well, and their accuracy is questionable. A primary reason for this, especially with Maximum Likelihood

Estimation (MLE) and Maximum Product Spacing (MPS), is a flat likelihood function with respect to ω. When the likelihood function is flat, small changes in the data can lead to large, unstable changes in the estimated parameter. This typically happens with parameters that have a less direct impact on the distribution's overall shape. While the bias decreases as the sample size increases, suggesting the estimators are still consistent, potential remedies include using Bayesian methods with informative priors to regularize the estimation or employing constrained optimization to ensure the parameter stays within its valid range, thus improving the stability and reliability of the results.

DATA AND CODE AVAILABILITY STATEMENT

Data are included in this manuscript and the R codes are accessed via https://github.com/obulezi12345-svg/R-codes-for-Arcsine-ratio-sine-generalized-family-of-distribution

Competing Interest

The authors declare that they have no competing interests.

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APPENDIX

A. PROOF OF IDENTIFIABILITY

Theorem 2 (Identifiability)**.** *A family of distributions is identifiable if different parameter values produce distinct distributions. For the ARS distribution, this means if $F(x, \theta_1, \zeta) = F(x, \theta_2, \zeta)$ for all x in the support of the random variable, then it necessarily implies $\theta_1 = \theta_2$.*

Proof. Let's assume that for two parameter values, θ_1 and θ_2, the CDFs are identical for all x in the domain of the distribution. That is, $F(x, \theta_1, \zeta) = F(x, \theta_2, \zeta)$ for all $x \in \mathbb{R}$.
Let $S(x;\zeta) = \sin\left(\frac{\pi}{2}G(x;\zeta)\right)$. As x varies over the support of the baseline distribution, $G(x;\zeta)$ ranges from 0 to 1, and thus $\frac{\pi}{2}G(x;\zeta)$ ranges from 0 to $\frac{\pi}{2}$. Consequently, $S(x;\zeta)$ can take on a continuum of values within [0, 1].
From the definition of the ARS CDF, we have:

$$1 - \frac{\theta_1(1 - S(x;\zeta))}{\theta_1 + (2 - \theta_1)S(x;\zeta)} = 1 - \frac{\theta_2(1 - S(x;\zeta))}{\theta_2 + (2 - \theta_2)S(x;\zeta)}$$

Subtracting 1 from both sides and multiplying by -1, we get:

$$\frac{\theta_1(1 - S(x;\zeta))}{\theta_1 + (2 - \theta_1)S(x;\zeta)} = \frac{\theta_2(1 - S(x;\zeta))}{\theta_2 + (2 - \theta_2)S(x;\zeta)}$$

Let $S = S(x;\zeta)$ for brevity.

$$\frac{\theta_1(1 - S)}{\theta_1 + (2 - \theta_1)S} = \frac{\theta_2(1 - S)}{\theta_2 + (2 - \theta_2)S}$$

We need to consider two cases for $(1 - S)$:

1. If $1 - S \neq 0$: (i.e., $S \neq 1$, which means $G(x;\zeta) \neq 1$) In this case, we can divide both sides by $(1 - S)$:

$$\frac{\theta_1}{\theta_1 + (2 - \theta_1)S} = \frac{\theta_2}{\theta_2 + (2 - \theta_2)S}$$

   Cross-multiplying gives:

$$\theta_1(\theta_2 + (2 - \theta_2)S) = \theta_2(\theta_1 + (2 - \theta_1)S)$$
$$\theta_1\theta_2 + \theta_1(2 - \theta_2)S = \theta_2\theta_1 + \theta_2(2 - \theta_1)S$$

   Subtracting $\theta_1\theta_2$ from both sides:

$$\theta_1(2 - \theta_2)S = \theta_2(2 - \theta_1)S$$

   Since this equality must hold for all x (meaning S can take on a continuum of values in [0, 1)), we can choose an x such that $S \neq 0$ (e.g., any x for which $G(x;\zeta) > 0$). Assuming $S \neq 0$:

$$\theta_1(2 - \theta_2) = \theta_2(2 - \theta_1)$$

$$2\theta_1 - \theta_1\theta_2 = 2\theta_2 - \theta_2\theta_1$$

   Adding $\theta_1\theta_2$ to both sides:

$$2\theta_1 = 2\theta_2$$
$$\theta_1 = \theta_2$$

2. If $1 - S = 0$: (i.e., $S = 1$, which means $G(x;\zeta) = 1$) In this case, the original equation becomes $F(x, \theta_1, \zeta) = 1$ and $F(x, \theta_2, \zeta) = 1$. This means both CDFs are 1 at values of x where the baseline CDF reaches 1. This condition is satisfied for any θ_1, θ_2 and does not provide information to distinguish them. However, since the previous case ($S \neq 1$) covers almost all x in the domain (except potentially a single point at the end of the support where $G(x;\zeta) = 1$), and the identity $\theta_1 = \theta_2$ was derived for that broad range, the identifiability holds.

   Specifically, the function $H(S) = \frac{\theta(1-S)}{\theta+(2-\theta)S}$ must be the same for all $S \in [0, 1]$. We have shown that if $H(S)$ is the same for $S \in [0, 1)$, then $\theta_1 = \theta_2$. The fact that $H(1) = 0$ for any θ is consistent with this.

Since the equality $F(x, \theta_1, \zeta) = F(x, \theta_2, \zeta)$ must hold for all x in the support of the random variable, and this implies that the function $\frac{\theta(1-S)}{\theta+(2-\theta)S}$ must be identical for all valid values of S, including a continuum of values where $S \neq 0$ and $S \neq 1$. The derived algebraic manipulation demonstrates that this is only possible if $\theta_1 = \theta_2$. Therefore, the ARS distribution family is identifiable with respect to its shape parameter θ. This completes the proof.

B. LINEAR EXPANSION OF THE ARS FAMILY

For clarity, let

$$(1+y)^{-n} = \sum_{i=0}^{\infty}(-1)^i\binom{n+i-1}{i}y^i;$$
$$\cos(z) = \sum_{j=0}^{\infty}(-1)^j\frac{(z)^{2j}}{(2j)!}; \tag{15}$$
$$\sin(z) = \sum_{k=0}^{\infty}(-1)^k\frac{(z)^{2k+1}}{(2k+1)!}.$$

$$\left(\sum_{v=0}^{\infty} b_v y^v\right)^s = \sum_{v=0}^{\infty} d_{v,s}y^v \tag{16}$$

where $d_{0,s} = b_0^s$ and for $v \geq 1$, the coefficients $d_{v,s}$ are recursively defined as: $d_{v,s} = (vb_0)^{-1}\sum_{p=1}^{v}[s(p-1)-v]b_p d_{v-p,s}$.

B.1 Derivation of the Linear Expansion for the ARS family PDF

Proof. The PDF of the ARS family is given by (2). Let $G = G(x;\zeta)$ be the baseline CDF and $Z = \frac{\pi}{2}G$. We rewrite the PDF as:

$$f(x;\theta,\zeta) = \frac{\pi\theta g(x;\zeta)\cos(Z)}{(\theta+(2-\theta)\sin(Z))^2} = \frac{\pi}{\theta}g(x;\zeta)\cos(Z)\left[1+\frac{2-\theta}{\theta}\sin(Z)\right]^{-2}$$

We apply the series expansions as follows:

1. Inverse square term expansion: Using $(1+y)^{-n}$ from (15) with $n=2$ and $y = \frac{2-\theta}{\theta}\sin(Z)$:

$$[1+\frac{2-\theta}{\theta}\sin(Z)]^{-2} = \sum_{i=0}^{\infty}(-1)^i(i+1)\left(\frac{2-\theta}{\theta}\right)^i(\sin(Z))^i$$

2. Trigonometric function expansions: From (15), with $Z = \frac{\pi}{2}G$:

$$\cos(Z) = \sum_{j=0}^{\infty}\frac{(-1)^j(\frac{\pi}{2})^{2j}}{(2j)!}G^{2j}$$
$$\sin(Z) = \sum_{k=0}^{\infty}\frac{(-1)^k(\frac{\pi}{2})^{2k+1}}{(2k+1)!}G^{2k+1}$$

Let $a_j = \frac{(-1)^j(\frac{\pi}{2})^{2j}}{(2j)!}$ and $c_k = \frac{(-1)^k(\frac{\pi}{2})^{2k+1}}{(2k+1)!}$. So, $\cos(Z) = \sum_{j=0}^{\infty}a_jG^{2j}$ and $\sin(Z) = \sum_{k=0}^{\infty}c_kG^{2k+1}$.

3. Power series of $\sin(Z)$: The term $(\sin(Z))^i = \left(\sum_{k=0}^{\infty}c_kG^{2k+1}\right)^i$ is a power series in G. To apply (16), we can write $\sin(Z) = G\sum_{k=0}^{\infty}c_k(G^2)^k$. Let $S^*(Y) = \sum_{k=0}^{\infty}c_kY^k$, where $Y = G^2$. The coefficient $c_0 = \frac{\pi}{2}$ is non-zero.

   Then $(\sin(Z))^i = G^i\left(S^*(G^2)\right)^i$. Applying (16) to $(S^*(G^2))^i$, we get $\left(S^*(G^2)\right)^i = \sum_{v=0}^{\infty}d^*_{v,i}(G^2)^v = \sum_{v=0}^{\infty}d^*_{v,i}G^{2v}$, where $d^*_{v,i}$ are the coefficients from the power series expansion. Therefore, $(\sin(Z))^i = \sum_{v=0}^{\infty}d^*_{v,i}G^{2v+i}$.

Substituting these expansions back into the PDF formula yields:

$$f(x;\theta,\zeta) = \frac{\pi}{\theta}g(x;\zeta)\left(\sum_{j=0}^{\infty}a_jG^{2j}\right)\left(\sum_{i=0}^{\infty}(-1)^i(i+1)\left(\frac{2-\theta}{\theta}\right)^i\left(\sum_{v=0}^{\infty}d^*_{v,i}G^{2v+i}\right)\right)$$

Multiplying these series and collecting terms with the same power of $G(x;\zeta)$, the PDF can be expressed as a linear combination of the powers of the baseline CDF multiplied by its PDF:

$$f(x;\theta,\zeta) = \sum_{m=0}^{\infty}\Psi_m g(x;\zeta)G(x;\zeta)^m$$

where Ψ_m are constants derived from the combined coefficients $a_j, c_k, d^*_{v,i}$, and terms from $(-1)^i(i+1)\left(\frac{2-\theta}{\theta}\right)^i$, for all combinations of indices j,i,v such that $2j+2v+i = m$. The explicit form of Ψ_m is complex and involves multiple summations.

B.2 Derivation of the Linear Expansion for the ARS family CDF

Proof. The CDF of the ARS family is given by (1). Let $G = G(x;\zeta)$ and $Z = \frac{\pi}{2}G$. We rewrite the CDF as:

$$F(x;\theta,\zeta) = 1 - \frac{\theta(1-\sin(Z))}{\theta + (2-\theta)\sin(Z)} = 1 - (1-\sin(Z))\left[1 + \frac{2-\theta}{\theta}\sin(Z)\right]^{-1}$$

We apply the series expansions:

1. Inverse term expansion: Using $(1+y)^{-n}$ from (15) with $n = 1$ and $y = \frac{2-\theta}{\theta}\sin(Z)$:

$$\left[1 + \frac{2-\theta}{\theta}\sin(Z)\right]^{-1} = \sum_{p=0}^{\infty}(-1)^p\left(\frac{2-\theta}{\theta}\right)^p(\sin(Z))^p$$

2. Substitute $\sin(Z)$ power series: As derived for the PDF, $(\sin(Z))^p = \sum_{v=0}^{\infty} d^*_{v,p}G^{2v+p}$.

Substituting these into the CDF formula:

$$F(x;\theta,\zeta) = 1 - \left(1 - \sum_{k=0}^{\infty} c_k G^{2k+1}\right)\left(\sum_{p=0}^{\infty}(-1)^p\left(\frac{2-\theta}{\theta}\right)^p \sum_{v=0}^{\infty} d^*_{v,p}G^{2v+p}\right)$$

Expanding the product of these series and collecting terms with the same power of $G(x;\zeta)$, the CDF can be expressed as a linear combination of powers of the baseline CDF:

$$F(x;\theta,\zeta) = \sum_{m=0}^{\infty}\Phi_m G(x;\zeta)^m$$

where Φ_m are constants derived from the combined coefficients $c_k, d^*_{v,p}$, and terms from $(-1)^p\left(\frac{2-\theta}{\theta}\right)^p$, for all combinations of indices k, p, v such that $2k+1$ and $2v+p$ contribute to the overall power m.

B.3 ARS family Order Statistics

The PDF of the r-th order statistic, $X_{(r)}$, from a sample of size n drawn from the ARS-G distribution, is given by:

$$f_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x;\theta,\zeta)]^{r-1}[1 - F(x;\theta,\zeta)]^{n-r}f(x;\theta,\zeta)$$

To derive its linear expansion, we substitute the series expansions of $F(x;\theta,\zeta)$ and $f(x;\theta,\zeta)$ derived above. Let $F(x) = \sum_{m=0}^{\infty}\Phi_m G(x)^m$ and $f(x) = \sum_{k=0}^{\infty}\Psi_k g(x)G(x)^k$.

1. Expand $[F(x)]^{r-1}$: Using (16), $[F(x)]^{r-1} = \left(\sum_{m=0}^{\infty}\Phi_m G(x)^m\right)^{r-1} = \sum_{j=0}^{\infty} u_{j,r-1}G(x)^j$, where $u_{j,r-1}$ are coefficients derived recursively from Φ_m.

2. Expand $[1-F(x)]^{n-r}$: Using the binomial theorem, $[1-F(x)]^{n-r} = \sum_{p=0}^{n-r}(-1)^p\binom{n-r}{p}[F(x)]^p$.

   For each term $[F(x)]^p$, we apply (16): $[F(x)]^p = \left(\sum_{m=0}^{\infty}\Phi_m G(x)^m\right)^p = \sum_{q=0}^{\infty} v_{q,p}G(x)^q$. Thus, $[1-F(x)]^{n-r} = \sum_{p=0}^{n-r}(-1)^p\binom{n-r}{p}\sum_{q=0}^{\infty} v_{q,p}G(x)^q = \sum_{q=0}^{\infty} w_{q,n-r}G(x)^q$, where $w_{q,n-r}$ are the combined coefficients.

Substituting these expanded forms back into the formula for $f_{(r)}(x)$:

$$f_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\sum_{j=0}^{\infty} u_{j,r-1}G(x)^j\right)\left(\sum_{q=0}^{\infty} w_{q,n-r}G(x)^q\right)\left(\sum_{k=0}^{\infty}\Psi_k g(x)G(x)^k\right)$$

Multiplying these three series and collecting terms with the same power of $G(x)$, the PDF of the r-th order statistic can be expressed as a simple linear combination:

$$f_{(r)}(x) = \sum_{s=0}^{\infty}\Xi_s g(x;\zeta)G(x;\zeta)^s$$

where Ξ_s are the overall constants, dependent on n, r, and the combined coefficients $u_{j,r-1}, w_{q,n-r}, \Psi_k$, for all combinations of indices j, q, k such that $j + q + k = s$.