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Research Article

Solving Flexural-Torsional Buckling Equations of Thin-Walled Columns using Stodola-Vianello Successive Iteration Method

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ABSTRACT

In this study, Timoshenko's equations for the generalized elastic thin-walled column buckling analysis was solved by the Stodola-Vianello iteration method (SVIM). The candidate problem which is a system of three coupled ordinary differential equations (ODEs) in three displacement variables was expressed using four successive integrations as a system of three coupled iteration equations at the *n*th buckling mode. Boundary conditions (BCs) were applied to obtain the constants of integration. General solutions were then obtained for simply supported BCs for the general cases of: (a) unsymmetrical cross-sections, (b) doubly symmetrical cross-sections and (c) singly symmetrical cross-sections about the *yy* axis and *zz* axis respectively. It was found that for the doubly symmetric cross sections, the buckling equations are uncoupled and the least of the critical buckling loads in flexural buckling about the *yy* and *zz* axes and torsional buckling determines the failure. It was also found that the buckling types are coupled for unsymmetrical cross-sections. However, for doubly symmetrical cross-sections about the *zz* axis, the flexural buckling in the *zz* direction is uncoupled while the flexural buckling about the *yy* direction is coupled with the torsional buckling. For singly symmetrical cross-sections about the *yy* axis, the flexural buckling in the *yy* direction is uncoupled while the flexural buckling about the *zz* axis is coupled with the torsional buckling. The critical buckling load is found as the smallest of the buckling loads obtained by solving the algebraic eigenvalue problem.

1. INTRODUCTION

Generalized beam theory (GBT) can provide very accurate and computationally efficient buckling solutions for thin-walled beams and columns. In GBT, the strain-displacement equations permit deformations of the cross-section in-plane and out-of-plane (warping) [1]. GBT breaks down the deformation of the thin-walled member into a linear combination of cross-sectional deformation modes and distortional modes [1]. Flexural-torsional buckling (FTB) is a type of instability in slender structures with open cross sections (like channels, angles, I-beams) under compression or flexure, where the structural member bends laterally and simultaneously twists about its shear center. FTB is marked by out-of-plane deformation, which reduced the load carrying capacity. FTB is presented by using braces or stiffer cross-sections. FTB occurs mainly in members with low torsional stiffness, such as members with open cross-sections, or non-symmetric profiles where the shear centers do not coincide with the centroid. It occurs when the axial compressive load or bending moment attains a critical value leading to instability in the members, and resulting in a simultaneous bending about the minor axis and twisting deformation. It is thus vital for Engineers to check for FTB to ensure structural stability, particularly for long unbraced members. This is done by a critical FTB load analysis. FTB also called lateral-torsional buckling (LTB) is the buckling which occurs over the length of a member in which the cross-section moves out of the plane of bending and simultaneously undergoes torsional displacements [2 – 5]. FTB is an important theme in research and design to avert premature failures with the attendant unpleasant consequences. FTB has been studied using a variety of methods, including approximate and classical methods by Wang et al [6], Alsayed [5], Zhu [7], Howlett [8] and Al-Sheikh [9]. Duan et al [10] presented a GBT based finite element method (GBT-FEM) for the linear buckling analysis of perforated thin-walled beams.

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They were able to extend the GBT to thin-walled beams with arbitrary shapes and perforations. Sahraei [11] used the finite element method to study LTB of beam-columns and plane frames. Ekström and Wesley [12] studied FTB of steel channel beams using FEM and presented parametric studies. Niki [13] developed "shear-deformable hybrid FEM for buckling analysis of composite thin-walled members". Mohan et al [14] studied the FTB of columns. Riley [15] used Newmark method for the FTB analysis of thin columns with varying cross-sectional geometry along the longitudinal axes. Ferretti [16] investigated FTB of uniformly compressed beam-like structures. Banerjee [17] studied the buckling analysis of axial-flexural coupled columns with transverse shear deformation included in the derivation of the governing domain equations. The work used energy principles to derive the governing domain equations and finite element methods to find the stiffness matrices. Parametric solutions were found for columns with thin-walled open square box cross-section with clamped-clamped, clamped-free and pinned-simply supported ends respectively. Peres et al [18] extended the generalized beam theory (GBT) to enable buckling (bifurcation) analyses of thin-walled structures with circular axis and no pre-twist. They obtained bifurcation eigenvalue problem by using the concept of linear buckling to the non-linear equilibrium equations, while incorporating the GBT strain-displacement assumptions. A displacement finite element is then developed for the GBT formulation. Ike et al [19] used the Fourier cosine series method to solve in closed form the generalized elastic thin walled column buckling problem for Dirichlet boundary conditions". In their work the origin of coordinates was assumed to be at the column midspan to allow for the satisfaction of BCs. Their work gave exact buckling load solutions. Ike [20] presented "energy formulation for the FTB of thin-walled column with open cross-section". The formulation used the principle of total potential energy functional minimizations via Euler-Lagrange differential equations to obtain the governing domain equation. Exact buckling solutions of the formulated problem were obtained for Dirichlet boundary conditions. Alkan [21] studied the FTB analysis of thin-walled columns having asymmetric open cross-sections. The work utilized the method of transfer matrix for formulating the eigenvalue problem. The iterative procedure was used to extract the least eigenvalues which reasonably agreed with exact solutions. Gizejowski et al [22] applied the energy minimization method to the FTB analysis for complex loads that could be treated as symmetric and asymmetric loads or combinations thereof. Hassan [23] investigated the FTB of timber structures carrying axial compressive forces and bending moments. Barszcz et al [24, 25] studied FTB for open-section beam-column and bisymmetric beams. Other significant studies on FTB were presented in [26] and [27]. A review of literature has shown that the Stodola-Vianello iteration method (SVIM) has not been explored for the solution of the Timoshenko's generalized elastic column buckling equations. This paper aims at using the SVIM to derive solutions to the Timoshenko's generalized elastic column buckling equations. The merit of the method is that the governing equation is transformed from a system of ODEs to a system of iteration equations. The inspiration for the application of SVIM to the candidate problem lies in the successful use of the SVIM in buckling problems by Ike [28, 29]

2. METHODOLOGY

2.1 Differential equations of flexural torsional buckling of beams

Timoshenko derived the differential equations of flexural torsional buckling (FTB) of beams as the system of three coupled Equations (1a, 1b, 1c), where $v(x)$, $w(x)$ and $\theta_x(x)$ are the unknown displacement variables [19] – [20].

$$EI_{zz}\frac{d^4v}{dx^4}+P_x\frac{d^2v}{dx^2}+P_xe_z\frac{d^2\theta_x}{dx^2}=0 \quad (1a)$$

$$EI_{yy}\frac{d^4w}{dx^4}+P_x\frac{d^2w}{dx^2}-P_xe_y\frac{d^2\theta_x}{dx^2}=0 \quad (1b)$$

$$EI_w\frac{d^4\theta_x}{dx^4}-\left(GJ-\frac{P_xI_0}{A}\right)\frac{d^2\theta_x}{dx^2}+P_xe_z\frac{d^2v}{dx^2}-P_xe_y\frac{d^2w}{dx^2}=0 \quad (1c)$$

wherein I_0 is the polar moment of inertia about the shear center, E is the Young's modulus of elasticity, G is the shear modulus or modulus of rigidity, J is the Saint Venant torsional stiffness of the cross-section, e_y, e_z are the coordinates of the shear center, A is the cross-sectional area, I_w is the warping constant, I_{zz} is the moment of inertia about z, I_{yy} is the moment of inertia about y, P_x is a longitudinal force applied to the cross section.

$$I_0=I_{yy}+I_{zz}+(e_y^2+e_z^2)A$$

$$\frac{I_0}{A}=r_0^2=e_y^2+e_z^2+\left(\frac{I_{xx}+I_{yy}}{A}\right) \quad (2)$$

r_0 is the radius of gyration.

2.2 Stodola-Vianello successive iteration equations (SVSIEs)

Dividing Equations (1a), (1b), (1c) respectively by EI_{zz}, EI_{yy} and EI_w respectively gives:

$$v^{iv}(x) + \frac{P_x}{EI_{zz}} v''(x) + \frac{P_x e_z}{EI_{zz}} \theta''_x(x) = 0 \tag{3a}$$

$$w^{iv}(x) + \frac{P_x}{EI_{yy}} w''(x) - \frac{P_x e_y}{EI_{yy}} \theta''_x = 0 \tag{3b}$$

$$\theta_x^{iv} - \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{AEI_w}\right)\theta''_x + \frac{P_x e_z}{EI_w} v''(x) - \frac{P_x e_y}{EI_w} w''(x) = 0 \tag{3c}$$

where the primes denote derivatives with respect to x.
Integrating Equation (3c) successively gives:

$$v'''(x) + \frac{P_x}{EI_{zz}} v'(x) + \frac{P_x e_z}{EI_{zz}} \theta'_x + c_1 = 0 \tag{4a}$$

where c_1 is an integration constant.

$$v''(x) + \frac{P_x}{EI_{zz}} v(x) + \frac{P_x e_z}{EI_{zz}} \theta_x(x) + c_1 x + c_2 = 0 \tag{4b}$$

c_2 is an integration constant.

$$v'(x) + \frac{P_x}{EI_{zz}} \int_0^x v(x)dx + \frac{P_x e_z}{EI_{zz}} \int_0^x \theta_x(x)dx + \frac{c_1 x^2}{2} + c_2 x + c_3 = 0 \tag{4c}$$

c_3 is an integration constant.

$$v(x) + \frac{P_x}{EI_{zz}} \int_0^x\int_0^x v(x)dxdx + \frac{P_x e_z}{EI_{zz}} \int_0^x\int_0^x \theta_x(x)dxdx + \frac{c_1 x^3}{3!} + \frac{c_2 x}{2!} + c_3 x + c_4 = 0 \tag{4d}$$

c_4 is an integration constant.
Hence, the iteration equations are:

$$v'''_{n+1}(x) = -\left(\frac{P_x}{EI_{zz}} v'_n(x) + \frac{P_x e_z}{EI_{zz}} \theta'_{xn}(x) + c_1\right) \tag{5a}$$

$$v''_{n+1}(x) = -\left(\frac{P_x}{EI_{zz}} v_n(x) + \frac{P_x e_z}{EI_{zz}} \theta_{xn}(x) + c_1 x + c_2\right) \tag{5b}$$

$$v'_{n+1}(x) = -\left(\frac{P_x}{EI_{zz}} \int_0^x v_n(x)dx + \frac{P_x e_z}{EI_{zz}} \int_0^x \theta_{xn}(x)dx + \frac{c_1 x^2}{2!} + c_2 x + c_3\right) \tag{5c}$$

$$v_{n+1}(x) = -\left(\frac{P_x}{EI_{zz}} \int_0^x\int_0^x v_{nx}(x)dxdx + \frac{P_x e_z}{EI_{zz}} \int_0^x\int_0^x \theta_{xn}(x)dxdx + \frac{c_1 x^3}{3!} + \frac{c_2 x^2}{2!} + c_3 x + c_4\right) \tag{5d}$$

Similarly, successive integrations of Equation (3a) give the iteration equations:

$$w'''_{n+1}(x) = -\left(\frac{P_x}{EI_{yy}} w'_n(x) - \frac{P_x e_y}{EI_{yy}} \theta'_{xn}(x) + c_5\right) \tag{6a}$$

c_5 is an integration constant.

$$w''_{n+1}(x) = -\left(\frac{P_x}{EI_{yy}} w_n(x) - \frac{P_x e_y}{EI_{yy}} \theta_{xn}(x) + c_5 x + c_6\right) \tag{6b}$$

c_6 is an integration constant.

$$w'_{n+1}(x) = -\left(\frac{P_x}{EI_{yy}} \int_0^x w_n(x)dx - \frac{P_x e_y}{EI_{yy}} \int_0^x \theta_{xn}(x)dx + \frac{c_5 x^2}{2!} + c_6 x + c_7\right) \tag{6c}$$

c_7 is an integration constant.

$$w_{n+1}(x) = -\left(\frac{P_x}{EI_{yy}}\int_0^x\int_0^x w_n(x)dxdx - \frac{P_xe_y}{EI_{yy}}\int_0^x\int_0^x \theta_{xn}(x)dxdx + \frac{c_5x^3}{3!} + \frac{c_6x^2}{2!} + c_7x + c_8\right) \quad (6d)$$

c_8 is an integration constant.

The SVSIEs for Equation (3c) are:

$$\theta'''_{x_{n+1}}(x) = -\left(-\left(\frac{GJ}{EI_w} - \frac{P_xI_0}{AEI_w}\right)\theta'_{x_n} + \frac{P_xe_z}{EI_w}v'_n(x) - \frac{P_xe_y}{EI_w}w'(x) + c_9\right) \quad (7a)$$

c_9 is an integration constant.

$$\theta''_{x_{n+1}}(x) = -\left(-\left(\frac{GJ}{EI_w} - \frac{P_xI_0}{AEI_w}\right)\theta_{x_n} + \frac{P_xe_z}{EI_w}v_n(x) - \frac{P_xe_y}{EI_w}w(x) + c_9x + c_{10}\right) \quad (7b)$$

c_{10} is an integration constant.

$$\theta'_{x_{n+1}}(x) = -\left(-\left(\frac{GJ}{EI_w} - \frac{P_xI_0}{AEI_w}\right)\int_0^x \theta_{x_n}(x)dx + \frac{P_xe_z}{EI_w}\int_0^x v_n(x)dx - \frac{P_xe_y}{EI_w}\int_0^x w_n(x)dx + \frac{c_9x^2}{2!} + c_{10}x + c_{11}\right) \quad (7c)$$

c_{11} is an integration constant.

$$\theta_{x_{n+1}}(x) = -\left(-\left(\frac{GJ}{EI_w} - \frac{P_xI_0}{AEI_w}\right)\int_0^x\int_0^x \theta_{x_n}(x)dxdx + \frac{P_xe_z}{EI_w}\int_0^x\int_0^x v_n(x)dxdx - \frac{P_xe_y}{EI_w}\int_0^x\int_0^x w_n(x)dxdx + \frac{c_9x^3}{3!} + \frac{c_{10}x^2}{2!} + c_{11}x + c_{12}\right) \quad (7d)$$

c_{12} is an integration constant.

Boundary conditions (BCs)

The boundary conditions for simply supported ends ($x = 0$, $x = l$) are

$$v(0) = 0,\ \ v(l) = 0,\ \ v''(0) = 0 = v''(l) = 0$$
$$w(0) = 0,\ \ w(l) = 0 = w''(0) = w''(l) = 0 \quad (8)$$
$$\theta_x(0) = 0,\ \ \theta_x(l) = 0 = \theta''_x(0) = \theta''_x(l) = 0$$

For beams clamped at $x = 0$ and $x = l$, the BCs are:

$$v(0) = v(l) = v'(0) = v'(l) = 0$$
$$w(0) = w(l) = w'(0) = w'(l) = 0 \quad (9)$$
$$\theta_x(0) = \theta_x(l) = \theta'_x(0) = \theta'_x(l) = 0$$

The SVSIEs are implemented by choosing the FTB mode shape functions that satisfy the BCs. For beams with simply supported BCs, the FTB mode functions at the nth mode are:

$$v_n(x) = B_{1n}\sin\frac{n\pi x}{l}$$
$$w_n(x) = B_{2n}\sin\frac{n\pi x}{l} \quad (10)$$
$$\theta_{x_n}(x) = B_{3n}\sin\frac{n\pi x}{l}$$

where B_{1n}, B_{2n}, B_{3n} are the nth modal amplitudes of $v_n(x)$, $w_n(x)$ and $\theta_{xn}(x)$ respectively.

Noting that

$$\int_0^x\int_0^x v_n(x)dxdx = \int_0^x\int_0^x B_{1n}\sin\frac{n\pi x}{l}dxdx = -B_{1n}\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l} \quad (11)$$

$$\int_0^x\int_0^x w_n(x)dxdx = \int_0^x\int_0^x B_{2n}\sin\frac{n\pi x}{l}dxdx = -B_{2n}\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l} \quad (12)$$

$$\int_0^x\int_0^x \theta_{x_n}(x)dxdx = \int_0^x\int_0^x B_{3n}\sin\frac{n\pi x}{l}dxdx = -B_{3n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} \quad (13)$$

The use of BCs give the integration constants as:

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{12} = 0 \quad (14)$$

Hence the iteration equations become from Equation (14),

$$v_{n+1}(x) = \frac{P_x}{EI_{zz}}\cdot B_{1n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} + \frac{P_x e_z}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 B_{3n}\sin\frac{n\pi x}{l} \quad (15)$$

$$w_{n+1}(x) = \frac{P_x}{EI_{yy}} B_{2n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - \frac{P_x e_y}{EI_{yy}} B_{3n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} \quad (16)$$

$$\theta_{x_{n+1}}(x) = -\left(\frac{GJ}{EI_w} - \frac{P_x I_0}{AEI_w}\right)B_{3n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} + \frac{P_x e_z}{EI_w}B_{1n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - \frac{P_x e_y}{EI_w}B_{2n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} \quad (17)$$

3. RESULTS AND DISCUSSION

At convergence of the iterations,

$$v_{n+1}(x) = v_n(x)$$
$$w_{n+1}(x) = w_n(x) \quad (18)$$
$$\theta_{x(n+1)}(x) = \theta_{x_n}(x)$$

The convergence rule yields the system of iterations given as:

$$B_{1n}\sin\frac{n\pi x}{l} = \frac{P_x}{EI_{zz}}B_{1n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} + \frac{P_x e_z}{EI_{zz}}B_{3n}\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} \quad (19)$$

$$B_{2n}\sin\frac{n\pi x}{l} = \frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 B_{2n}\sin\frac{n\pi x}{l} - \frac{P_x e_y}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 B_{3n}\sin\frac{n\pi x}{l} \quad (20)$$

$$B_{3n}\sin\frac{n\pi x}{l} = -\left(\frac{GJ}{EI_w} - \frac{P_x I_0}{AEI_w}\right)\left(\frac{l}{n\pi}\right)^2 B_{3n}\sin\frac{n\pi x}{l} + \frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2 B_{1n}\sin\frac{n\pi x}{l} - \frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 B_{2n}\sin\frac{n\pi x}{l} \quad (21)$$

The system of equations are simplified as:

$$\left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1\right)B_{1n} + \frac{P_x e_z}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 B_{3n} = 0 \quad (22)$$

$$\left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 - 1\right)B_{2n} - \frac{P_x e_y}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 B_{3n} = 0 \quad (23)$$

$$\left(-1 - \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right)B_{3n} + \frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2 B_{1n} - \frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 B_{2n} = 0 \quad (24a)$$

Or,

$$-\frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2 B_{1n} + \frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 B_{2n} + \left(1 + \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right)B_{3n} = 0 \quad (24b)$$

In matrix form,

$$\begin{pmatrix} \left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2-1\right) & 0 & \frac{P_x e_z}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 \\ 0 & \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2-1\right) & -\frac{P_x e_y}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 \\ -\frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1+\left(\frac{GJ}{EI_w}-\frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \end{pmatrix}\begin{pmatrix} B_{1n} \\ B_{2n} \\ B_{3n} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (25)$$

Case 1: Unsymmetrical cross-section $e_y \neq 0, \; e_z \neq 0$

The buckling equation is found from the vanishing of the coefficient matrix as:

$$\begin{vmatrix} \left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2-1\right) & 0 & \frac{P_x e_z}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 \\ 0 & \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2-1\right) & -\frac{P_x e_y}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 \\ -\frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1+\left(\frac{GJ}{EI_w}-\frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \end{vmatrix}=0$$

The roots of the resulting polynomial in P_x yields the buckling load and the least root when $n = 1$ governs the buckling. The flexural and torsional buckling cases are coupled.

Case 2: Doubly symmetric cross-sections

Here, $e_y = e_z = 0$. When $e_y = e_z = 0$, the equations become:

$$\begin{pmatrix} \left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2-1\right) & 0 & 0 \\ 0 & \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2-1\right) & 0 \\ 0 & 0 & \left(1+\left(\frac{GJ}{EI_w}-\frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \end{pmatrix}\begin{pmatrix} B_{1n} \\ B_{2n} \\ B_{3n} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (26)$$

For non-trivial solutions,

$$\begin{vmatrix} \left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2-1\right) & 0 & 0 \\ 0 & \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2-1\right) & 0 \\ 0 & 0 & \left(1+\left(\frac{GJ}{EI_w}-\frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \end{vmatrix}=0 \quad (27)$$

Expanding the determinant gives:

$$\left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2-1\right)\left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2-1\right)\left(1+\left(\frac{GJ}{EI_w}-\frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right)=0 \quad (28)$$

The buckling equations are uncoupled when $e_z = e_y = 0$, and the load acts at the centroid of the cross-section. The uncoupled buckling equations are:

$$\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1 = 0$$

$$\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 - 1 = 0 \tag{29}$$

$$1 + \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2 = 0$$

The uncoupled eigenvalues are found by solving for P_x in each of the equations.

$$P_x = EI_{zz}\left(\frac{n\pi}{l}\right)^2 = Q_{zz(\text{Euler})}$$

The critical buckling stress in the z direction is $\sigma_{zz_{cr}}$ and is:

$$\sigma_{zz_{cr}} = \frac{Q_{zz(n=1)}}{A} = \frac{\pi^2 EI_{zz}}{l^2 A} = \frac{\pi^2 E r_{zz}^2}{l^2} \tag{30}$$

$$r_{zz}^2 = \frac{I_{zz}}{A}$$

where $Q_{zz(\text{Euler})}$ is the Euler nth buckling load about the zz axis, r_{zz} is the radius of gyration in the z direction.

$$P_x = EI_{yy}\left(\frac{n\pi}{l}\right)^2 = Q_{yy(\text{Euler})}$$

$$\sigma_{yy_{cr}} = \frac{Q_{yy(n=1)}}{A} = \frac{\pi^2 EI_{yy}}{l^2 A} = \frac{\pi^2 E r_{yy}^2}{l^2} \tag{31}$$

where $r_{yy}^2 = \frac{I_{yy}}{A}$

$\sigma_{yy_{cr}}$ is the critical buckling stress in the y direction, r_{yy} is the radius of gyration in the y direction, $Q_{yy(\text{Euler})}$ is the Euler nth buckling load about the yy axis.

$$\left(\frac{GJ}{EI_w} - \frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2 = -1 \tag{32}$$

$$\frac{1}{EI_w}\left(GJ - \frac{P_x I_0}{A}\right)\left(\frac{l}{n\pi}\right)^2 = -1$$

$$\left(GJ - \frac{P_x I_0}{A}\right)\left(\frac{l}{n\pi}\right)^2 = -EI_w$$

$$GJ - \frac{P_x I_0}{A} = -EI_w\left(\frac{n\pi}{l}\right)^2$$

$$\left(\frac{P_x I_0}{A} - GJ\right) = EI_w\left(\frac{n\pi}{l}\right)^2$$

$$\frac{P_x I_0}{A} = GJ + EI_w\left(\frac{n\pi}{l}\right)^2$$

$$P_x = \frac{A}{I_0}\left(GJ + EI_w\left(\frac{n\pi}{l}\right)^2\right) = Q_t \tag{33}$$

Q_t is the twist or torsional buckling load in the nth buckling mode.

Hence, $\frac{P_x}{A} = \frac{Q_t}{A} = \frac{1}{I_0}\left(GJ + EI_w\left(\frac{n\pi}{l}\right)^2\right) = \sigma_t$

σ_t is the torsional stress for the *n*th buckling mode.

The critical torsional buckling stress, $\sigma_{t_{cr}}$, is:

$$\sigma_{t_{cr}} = \sigma_t(n=1) = \frac{1}{I_0}\left(GJ + EI_w \frac{\pi^2}{l^2}\right) \tag{33a}$$

Case 3: Singly symmetrical cross-sections with $(e_y = 0,\ e_z \neq 0)$.

The buckling equations become in matrix form:

$$\begin{vmatrix} \left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1\right) & 0 & \frac{P_x e_z}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 \\ 0 & \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 - 1\right) & 0 \\ -\frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2 & 0 & \left(1 + \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{AEI_w}\right)\left(\frac{l}{n\pi}\right)^2\right) \end{vmatrix} = 0 \tag{34}$$

Expanding gives:

$$\left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 - 1\right)\begin{vmatrix} \left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1\right) & \frac{P_x e_z}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 \\ -\frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1 + \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \end{vmatrix} = 0 \tag{35}$$

Hence,

$$\left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 - 1\right)\left\{\left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1\right)\left(1 + \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) - \left(\frac{P_x e_z}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2\right)\left(-\frac{P_x e_z}{EI_w}\left(\frac{l}{n\pi}\right)^2\right)\right\} = 0 \tag{36}$$

$$\left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 - 1\right)\left\{\left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1\right)\left(1 + \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{AEI_w}\right)\left(\frac{l}{n\pi}\right)^2\right) + \frac{P_x^2 e_z^2}{EI_{zz}EI_w}\left(\frac{l}{n\pi}\right)^4\right\} = 0 \tag{37}$$

Case 4: Singly symmetric cross-section with $e_z = 0$

For $e_z = 0,\ e_y \neq 0$, the stability matrix equation becomes:

$$\begin{pmatrix} \left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1\right) & 0 & 0 \\ 0 & \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 - 1\right) & -\frac{P_x e_y}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 \\ 0 & -\frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1 + \left(\frac{GJ}{EI_w} - \frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \end{pmatrix}\begin{pmatrix} B_{1n} \\ B_{2n} \\ B_{3n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{38}$$

For $e_z = 0,\ e_y \neq 0$, the equations are uncoupled and simplify to:

$$\left(\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1\right)(B_{1n}) = 0 \tag{39}$$

$$\begin{pmatrix} -\frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1+\left(\frac{GJ}{EI_w}-\frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \\ \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2-1\right) & -\frac{P_x e_y}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 \end{pmatrix}\begin{pmatrix} B_{2n} \\ B_{3n} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (40)$$

The buckling equations become:

$$\frac{P_x}{EI_{zz}}\left(\frac{l}{n\pi}\right)^2 - 1 = 0$$

$$\therefore\ P_x = EI_{zz}\left(\frac{n\pi}{l}\right)^2 = (n\pi)^2\frac{EI_{zz}}{l^2}$$

$$P_x = Q_{zz(\text{Euler})}$$

where $Q_{zz(\text{Euler})}$ is the Euler buckling load for the nth mode.
The coupled buckling mode equation is found by solving:

$$\begin{vmatrix} -\frac{P_x e_y}{EI_w}\left(\frac{l}{n\pi}\right)^2 & \left(1+\left(\frac{GJ}{EI_w}-\frac{P_x I_0}{EI_w A}\right)\left(\frac{l}{n\pi}\right)^2\right) \\ \left(\frac{P_x}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2-1\right) & -\frac{P_x e_y}{EI_{yy}}\left(\frac{l}{n\pi}\right)^2 \end{vmatrix} = 0 \quad (41)$$

Parametric study

The cross-section of a bisymmetric column is shown in Figure 1. In the figure, t = 7mm, h = 150mm, E = 210,000N/mm^2, G = 80,000N/mm^2. The column is pinned at both ends. Determine the length of the column for torsional buckling to control the design.

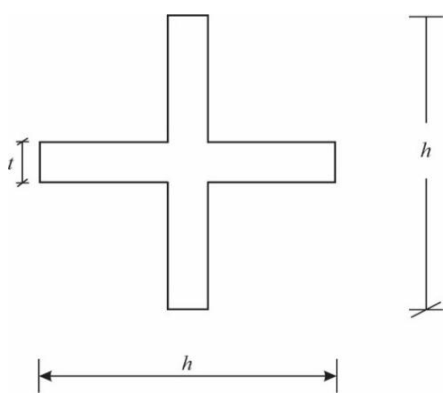

Figure 1: Cross-section of bisymmetrical column

Solution:

The section properties I_{yy}, and I_{zz} are:

$$I_{yy} = \frac{th^3}{12} + \frac{ht^3}{12} = \frac{th(h^2+t^2)}{12} = I_{zz}$$

$$I_{yy} = I_{zz} = \frac{7\times 150(7^2+150^2)}{12}\text{mm}^4$$

$$I_{yy} = I_{zz} = 1{,}973{,}037.5\text{mm}^4$$

$$I_0 = I_{yy} + I_{zz} = 3{,}946{,}075\text{mm}^4$$

$$A = 2th - t^2 = (2 \times 7 \times 150 - 7^2)\text{mm}^2$$

$$A = 2051\text{mm}^2$$

$$I_w = \frac{t^3 h^3}{72} = \frac{7^3 \times 150^3}{72}\text{mm}^6$$

$$I_w = 16,078,125\text{mm}^6$$

$$J = \frac{4t^3(h/2)}{3} = \frac{2ht^3}{3} = \frac{2 \times 150 \times 7^3}{3}$$

$$J = 34,300\text{mm}^4$$

$$\sigma_{yy}^{\text{Euler}} = \frac{\pi^2 E}{\left(l/r_{yy}\right)^2}$$

$$r_{yy} = \sqrt{\frac{I_{yy}}{A}}; \quad r_{zz} = \sqrt{\frac{I_{zz}}{A}}$$

$$r_{yy} = \sqrt{\frac{1973037.5}{2051}} = 31.016\text{mm} = r_{zz}$$

$$\sigma_{yy}^{\text{Euler}} = \frac{\pi^2 \cdot 210,000 \times 31.016^2}{l^2} = \sigma_{zz}^{\text{Euler}}$$

$$\sigma_{yy}^{\text{Euler}} = 2,020.18\left(\pi^2 / l^2\right) \times 10^5$$

For torsional buckling,

$$\sigma_t = \frac{1}{I_0}\left(EI_w \frac{\pi^2}{l^2} + GJ\right)$$

$$\sigma_t = \frac{1}{3,946,075}\left(\frac{210,000 \times 16,078,125\pi^2}{l^2} + 80,000 \times 34,300\right)$$

$$\sigma_t = \frac{1}{3.946 \times 10^6}\left(\frac{2.1 \times 10^5 \times 16.078 \times 10^6 \pi^2}{l^2} + 8 \times 10^4 \times 3.4 \times 10^4\right)$$

$$\sigma_t = \frac{33.7638 \times 10^{11} \pi^2}{3.946 \times 10^6 l^2} + \frac{27.2 \times 10^8}{3.946 \times 10^6}$$

$$\sigma_t = \frac{8.55646 \times 10^5 \pi^2}{l^2} + 6.893 \times 10^2$$

Torsion controls the design if

$$\sigma_{t_{cr}} < \sigma_{yy_{cr}}^{\text{Euler}}$$

$$\therefore \quad \frac{8.55646 \times 10^5 \pi^2}{l^2} + 6.893 \times 10^2 < \frac{2,021.18\pi^2 \times 10^5}{l^2}$$

$$2012.62354 \times 10^5 \frac{\pi^2}{l^2} > 6.893 \times 10^2$$

$$l^2 < \frac{2012.62354\pi^2 \times 10^5}{6.893 \times 10^2}$$

$$l^2 < 2,881,734.825\text{mm}^2$$

$$l < 1.697.567\text{mm}$$

$$l < 1.698\text{m}$$

It is observed that the *GJ* term dominates the torsional buckling equation unless the column is very short. For symmetrical cross-sections, the torsional buckling tends to control the design only for short columns.

4. CONCLUSION

In this paper, the Stodola-Vianello iteration method (SVIM) has been used to solve the Timoshenko equations for the flexural-torsional buckling of columns. The buckling equations were obtained as a system of iteration equations by successive integration. The exact solutions were found for simply supported ends using exact sinusoidal buckling functions. It can be concluded that:

1. In doubly symmetric cross-sections, the Timoshenko equations become decoupled in their buckling modes, and this yields characteristic buckling equations and buckling loads that are decoupled. The buckling types are flexural Euler buckling about the *yy* axis; flexural Euler buckling about the *zz* axis; and torsional buckling.
2. The least critical buckling load in the case of doubly symmetric cross-sections would determine the mode of failure.
3. In singly symmetric columns, with axis of symmetry as the *zz* axis, the flexural buckling mode in the *yy* axis is uncoupled, and the torsional buckling is coupled with the flexural buckling in the *zz* axis. The least critical buckling load of the two cases would govern the critical buckling load determination.
4. In the Timoshenko buckling problem where there is no symmetry about any axis, the flexural buckling modes about *yy* and *zz* axes and the torsional buckling modes are coupled and interact with one another.

Conflicts of Interest

The author declares no conflicts of interest.

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